Exercise 2

Physical constants

$M_\odot = 2\times10^{30}$ kg $\quad R_\odot = 7\times10^{8}$ m $\quad M_{\rm Sun} = 4.63$ (absolute magnitude) $\quad L_\odot = 3.83\times10^{26}$ J s^{-1}
1 AU $=1.5\times10^{11}$ m $\quad G = 6.67\times10^{-11}$ N m^2 kg^{-2} $\quad \sigma = 5.7\times10^{-8}$ kg s^{-3} K^{-4} (S-B constant)
$k_{\rm B} = 1.38\times10^{-23}$ m^2 kg s^{-2} K^{-1} $\quad m_{\rm H} = 1.67\times10^{-27}$ kg

Assessed questions

Question 1

(i) Consider a plasma composed of fully ionised hydrogen and helium only. From first principles, show that the mean molecular weight is given by

$$\mu = \frac{4}{(3+5X).}$$

Pressure is sum of partial pressures

$$P = \sum_i P_i = \sum_i n_i k_{\rm B} T.$$

Number of atoms of element i per unit volume is given by

$$\frac{\rho X_i}{A_i m_{\rm H}}$$

where ρ is the density, X_i is the mass fraction of element i, A_i is the atomic mass and $m_{\rm H}$ is the mass of the hydrogen atom atom. For fully ionised gas, total number of particles per unit volume for element i is given by

$$\frac{(Z_i+1)\rho X_i}{A_i m_{\rm H}}.$$

Hence we have

$$P = \sum_i \frac{(Z_i+1)X_i\rho}{A_i m_{\rm H}} k_{\rm B}T = \frac{k_{\rm B}}{\mu m_{\rm H}}\rho T$$

and thus

$$\mu^{-1} = \sum_i \frac{(Z_i+1)X_i}{A_i}.$$

For hydrogen we have $(Z_1+1)/A_1 = 2$ and for helium we have $(Z_2+1)/A_2 = 3/4$. We also have that $X_1 + X_2 = X + Y = 1$ for a gas composed of H and He only. Hence we can write

$$\mu^{-1} = 2X + \frac{3}{4}Y = 2X + \frac{3}{4}(1-X) = \frac{5X+3}{4}.$$

Therefore we have

$$\mu = \frac{4}{3+5X}.$$

20 marks

(ii) Calculate the value of μ if this gas has a helium mass fraction of 30%.
Substituting $X = 7/10$ into above expression gives $\mu = 8/13$. 10 marks
(iii) Consider an evolved star containing equal masses of fully ionised hydrogen and ^{12}C only. Calculate the value of μ for this stellar matter.
We have

$$\mu^{-1} = \sum_i \frac{(Z_i+1)X_i}{A_i}.$$

For hydrogen we have $(Z_1 + 1)/A_1 = 2$ and for ^{12}C we have $(Z_6 + 1)/A_6 = 7/12$, giving

$$\mu^{-1} = \frac{24X + 7Z}{12}$$

where we have denoted $X = X_1$ and $Z = X_6$, the mass fractions of H and ^{12}C, respectively. Hence we have

$$\mu = \frac{12}{24X + 7Z}.$$

Inserting $X = Z = 1/2$ gives $\mu = 24/31$. 15 marks

Question 2

Consider a uniform density star, composed of an ideal gas, that is in hydrostatic equilibrium. The density is given by $\rho = 3M/(4\pi R^3)$, where M is the stellar mass and R is the stellar radius.

(i) Obtain an expression for the gravitational acceleration at some arbitrary radius r inside the star.

For the gravitational acceleration we have

$$g = -\frac{Gm(r)}{r^2},$$

where

$$m(r) = 4\pi \int_0^r r^2 \rho dr.$$

For uniform density we obtain

$$g = -\frac{4}{3}\pi G\rho r.$$

10 marks

(ii) Show that the central pressure is given by

$$P_{\rm c} = \frac{3GM^2}{8\pi R^4}.$$

Hydrostatic equilibrium gives

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{4}{3}\pi G\rho r.$$

Integrating

$$\int_{P_{\rm c}}^0 dP = -\frac{4}{3}\pi G\rho \int_0^R r dr$$

gives

$$-P_{\rm c} = -\frac{4}{3}\pi G\rho\frac{R^2}{2}.$$

Substituting $\rho = 3M/(4\pi R^3)$ gives

$$P_{\rm c} = \frac{3M^2}{8\pi R^4}.$$

20 marks

(iii) Show that the central temperature is given by

$$T_{\rm c} = \frac{GM}{2R}\frac{\mu m_{\rm H}}{k_{\rm B}}.$$

The equation of state is

$$P_{\rm c} = \frac{k_{\rm B}}{\mu m_{\rm H}}\rho_{\rm c} T_{\rm c}$$

and rearranging gives

$$T_{\rm c} = \frac{P_{\rm c}}{\rho_{\rm c}}\frac{\mu m_{\rm H}}{k_{\rm B}}.$$

Noting that $\rho = 3M/(4\pi R^3)$, and using the equation for P_c derived above, we obtain

$$T_c = \frac{GM}{2R}\frac{\mu m_H}{k_B}.$$

15 marks

(iv) For a star of Solar mass and radius, composed of 30% helium and 70% hydrogen by mass, where both constituents are fully ionised, calculate the central temperature and pressure.

From above we know that $\mu = 8/13$. Inserting values into our equations gives $T_c \approx 7 \times 10^6$ K and $P_c \approx 1.3 \times 10^{14}$ N m^2. 10 marks

Non-assessed questions

(i) Make sure that you are able to follow the derivation of the mean molecular weight and the equation of hydrostatic equilibrium in detail, as given in the online lecture notes.

(ii) Make sure also that you are able to follow and reproduce the derivation of the virial theorem, and be able to demonstrate that for a star in hydrostatic equilibrium without internal sources of energy such as fusion reactions, half of the change in gravitational potential energy goes into increasing the internal energy, and the other half is lost through radiation.