Queen Mary
University of London

MSc/MSci Examination

Main Examination Period 2018

SPA7023P/SPA7023U Stellar Structure and Evolution Duration: 2 hours 30 minutes

YOU ARE NOT PERMITTED TO READ THE CONTENTS OF THIS QUESTION PAPER UNTIL INSTRUCTED TO DO SO BY AN INVIGILATOR.

Instructions:

Answer ALL questions from Section A. Answer ONLY TWO questions from Section B. Section A carries 50 marks, each question in section B carries 25 marks.

If you answer more questions than specified, only the first answers (up to the specified number) will be marked. Cross out any answers that you do not wish to be marked.

Only non-programmable calculators are permitted in this examination. Please state on your answer book the name and type of machine used.

Complete all rough workings in the answer book and cross through any work that is not to be assessed.

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EXAM PAPERS MUST NOT BE REMOVED FROM THE EXAM ROOM.

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You may assume the following:

The symbols used in all questions have the following meaning:

M is the stellar mass	$m(r)$ is the mass interior to radius r
R is the stellar radius	L the luminosity
T_{eff} the effective temperature of a star	P is the pressure
ρ is the density	T is the temperature
κ is the opacity per unit mass	ϵ the rate of energy production per unit mass
μ denotes the mean molecular weight	γ is the ratio of specific heats
$\mathcal{R}$ is the gas constant	c is the speed of light
G is the gravitational constant	σ is the Stefan-Boltzmann constant
a is Stefan's radiation constant	X, Y, Z are the mass fractions of H, He and heavy elements

Note the relation $\sigma = ac/4$.

You may assume $L = 4\pi R^2 F_{\mathrm{rad}}$, and F_{rad} is given by

$$F_{\mathrm{rad}} = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{dT}{dr}.$$

The central density, ρ_{c}, central temperature, T_{c}, and central pressure, P_{c}, of a polytrope of index n are:

$$\rho_{\mathrm{c}} = a_n\frac{3M}{4\pi R^3}, \qquad T_{\mathrm{c}} = b_n\frac{\mu GM}{\mathcal{R}R}, \qquad P_{\mathrm{c}} = c_n\frac{GM^2}{R^4},$$

where a_n, b_n and c_n are constants.

The apparent magnitude, m_{app}, absolute magnitude, M_{abs} and distance in parsecs are related by $m_{\mathrm{app}} = M_{\mathrm{abs}} + 5\log_{10} d - 5$.

The following rounded numerical values, all in S.I. units, may be assumed throughout the paper.

$$c = 3\times10^{8},\ G = 7\times10^{-11},\ \sigma = 6\times10^{-8},\ M_\odot = 2\times10^{30},\ R_\odot = 7\times10^{8},\ L_\odot = 4\times10^{26}.$$

You may also assume that 1 year is 3×10^{7} seconds.

SECTION A **Answer ALL questions in Section A**

Question A1

A CCD image of a region of the sky containing stars S_1 and S_2 was obtained. Star S_1 is known to have an absolute magnitude of $M_1 = 12$ and a parallax of 0.01 arcsec.
(i) State the apparent magnitude of star S_1.
(ii) S_1 gave a photon count of 12500 and S_2 a photon count of 25000 during the same time interval. State the apparent magnitude of S_2.
(iii) Both stars lie on the main sequence in the H-R diagram, which may be assumed to be a straight line of slope 5. The effective temperature of S_2 is twice that of S_1. State the absolute magnitude of S_2.

[10 marks]

Question A2

(i) Show that for a fully ionised gas consisting of atomic hydrogen, helium and $^{16}_{8}$O only, the mean molecular weight, μ, is given by

$$\mu = \frac{16}{20X + 12 - 3Z}.$$

(ii) After arriving at the main sequence, a star consists of H, He and $^{16}_{8}$O, and is of uniform chemical composition with $X = 0.70$ and $Y = 0.28$. As the star evolves, hydrogen is converted to helium, so that X and Y change. Calculate the change in μ between the initial state and when Y has increased to 0.35.

[10 marks]

Question A3

Consider a group of homogeneous stars. Each star in the group has the same chemical composition and is composed of an ideal gas. Energy is generated by the CNO cycle with $\epsilon = \epsilon_{\mathsf{CNO}}\rho T^{17}$, where ϵ_{CNO} is a constant. All energy is transported by radiation, and the main opacity is due to electron scattering so that κ is constant.
(i) Show that

$$R \propto M^{0.8}.$$

(ii) Also show that

$$L \propto M^3.$$

(iii) Obtain the slope of the line in an H-R diagram ($\log_{10} L$ versus $\log_{10} T_{\mathsf{eff}}$) that these stars lie on.

[10 marks]

Question A4

The temperature profile, $T(r)$, in a polytropic star composed of an ideal gas is

$$T(r) = T_{\text{c}}\theta(\xi),$$

where T_{c} is the central temperature, $\xi = r/\alpha$ where α is a constant, and $\theta(\xi)$ is the solution to the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

with boundary conditions $\theta(0) = 1$ and $d\theta/d\xi = 0$ at $\xi = 0$. The constant n is the polytropic index. Consider a polytropic star with $n = 0$.

(i) Show by direct integration of the Lane-Emden equation that

$$\theta(\xi) = 1 - \frac{\xi^2}{6}.$$

(ii) Deduce the value of ξ at the surface of the star.
(iii) The central temperature of the star is $T_{\text{c}} = 1 \times 10^8$ K. Find the temperature at a distance from the centre corresponding to 40% of the stellar radius.

[10 marks]

Question A5

The gravitational binding energy, Ω, of a spherically symmetric star is given by

$$\Omega = -G\int_0^M \frac{m(r)\,dm}{r},$$

where $m(r)$ is the mass interior to radius r.

(i) Consider a stellar model with uniform density (ρ = constant). Show that the gravitational binding energy of a star of mass M and radius R is given by

$$\Omega = -\frac{3}{5}\frac{GM^2}{R}.$$

(ii) Using the equation of hydrostatic equilibrium, $dP/dr = -Gm(r)\rho(r)/r^2$, show that the pressure profile, $P(r)$, inside the star of uniform density is

$$P(r) = \frac{3GM^2}{8\pi R^6}(R^2 - r^2).$$

(iii) Show that

$$\Omega = -3\int_V PdV,$$

where V is the volume of the star.

[10 marks]

SECTION B **Answer TWO questions from Section B**

Question B1

a) By considering the forces acting on a volume element, show that for a spherically symmetric star to be in hydrostatic equilibrium the following expression must apply:

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2}.$$

[6 marks]

Consider a hypothetical star of mass M and radius R, for which the density profile is represented by $\rho(r) = \rho_{\mathsf{c}}\,(1 - r/R)$.

b) Show that $m(r)$, the mass interior to r, can be written as

$$m(r) = \frac{4}{3}\pi r^3 \rho_{\mathsf{c}} \left(1 - \frac{3r}{4R}\right).$$

[3 marks]

c) Show that the mean density of the star is given by $\rho_{\mathsf{c}}/4$.

[2 marks]

d) Given that the pressure is zero when $r = R$, show that the central P_{c} is given by:

$$P_{\mathsf{c}} = \frac{5GM^2}{4\pi R^4}.$$

[5 marks]

e) The star is composed of an ideal gas with equation of state $P = \frac{\mathcal{R}}{\mu}\rho T$. Show that the central temperature is

$$T_{\mathsf{c}} = \frac{5\mu}{12\mathcal{R}}\frac{GM}{R}.$$

[2 marks]

f) The energy generation rate per unit mass for the PP-chain is given by $\epsilon_{\mathsf{pp}} = 3 \times 10^{-37} X^2 \rho T^4$ J Kg^{-1} s^{-1}, and that generated by the CNO-cycle is given by $\epsilon_{\mathsf{CNO}} = 8 \times 10^{-118} XZ\rho T^{16}$ J Kg^{-1} s^{-1}. Assuming that the mass-radius relation for the hypothetical star described above obeys the relation $R_* = R_\odot (M_*/M_\odot)^{3/4}$, determine the mass of the star for which the energy generation rate per unit mass at its centre has equal contributions from the PP-chain and the CNO cycle. You should assume that $\mathcal{R}/\mu = 10^4$, and that the mass fractions of hydrogen and heavy elements are $X = 0.98$ and $Z = 0.02$, respectively.

[7 marks]

Question B2

a) A star is in hydrostatic equilibrium,

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho(r),$$

where $m(r)$ is the mass interior to radius r. The gravitational binding energy is defined by

$$\Omega = -G\int_0^M \frac{m\, dm}{r}.$$

Using integration by parts, show that

$$\Omega = -3\int_V P dV,$$

where V is the spherical volume occupied by the star.

[8 marks]

b) The star is composed of classical particles. For a gas of classical particles, pressure P and internal energy per unit volume u are related by $u = 3P/2$, a relation that you can use without proof. Show that

$$\Omega = -2U,$$

where U is the total thermal energy of the star. Write down the total energy of the star. Before arriving at the main sequence, the star is contracting slowly under its own gravity. Show that half of the gravitational energy that is released during the contraction is radiated away, and the other half goes to increase the internal energy of the gas.

[5 marks]

c) The star of mass M contracts due to gravity, deriving its energy only from the change in its gravitational binding energy. The energy is transported outwards by convection, so that the star can be described by a polytropic model with polytropic index $n = 3/2$. The gravitational binding energy for a star of radius R and polytropic index n is given by

$$\Omega = -\frac{3GM^2}{(5-n)R}.$$

The virial theorem holds, and so you can assume that half the gravitational energy released is radiated away. Hence, show that the luminosity of the star at time t is

$$L = -\frac{3}{7}\frac{GM^2}{R^2(t)}\frac{dR}{dt}.$$

[4 marks]

d) The star may be assumed to evolve with effective temperature, T_{eff}, remaining constant. Show that the time, t_1, taken by such a star to evolve from a large radius to some smaller radius, R_1, is given by

$$t_1 = \frac{GM^2}{7L_1R_1},$$

where L_1 is the luminosity when the star has radius R_1.

[8 marks]

Question B3

According to Pauli's exclusion principle, at most two electrons can occupy a given energy state, and each particular energy state occupies volume h^3 in the 6-dimensional space of coordinates and momenta, where h is Planck's constant.

a) In a degenerate gas all the electron states are filled up to a threshold momentum, p_F, and none above this value are occupied. Show that the number density of electrons for which the momentum p is in the interval $(p, p + dp)$ is

$$n_e(p)dp = \frac{8\pi p^2}{h^3}dp,$$

when $p \leq p_\mathsf{F}$. Show that the total electron number density (i.e. integrated over all possible momenta), is

$$n_e = \frac{8\pi}{3h^3}p_\mathsf{F}^3.$$

[5 marks]

b) Show that when the pressure P is dominated by the electron pressure, and the electrons are moving with speeds comparable to the speed of light, c,

$$P = \frac{2\pi c}{3h^3}p_\mathsf{F}^4.$$

You may use the general expression

$$P = \frac{1}{3}\int_0^\infty v\ p\ n(p)\ dp,$$

where $n(p)\ dp$ is the number density of particles with momentum in the interval $(p, p + dp)$, and v is their velocity.

[4 marks]

c) Show that in a completely ionised gas of a given chemical composition, the electron number density, n_e, is proportional to the mass density, ρ.

[4 marks]

d) Show that in a stellar core where the pressure P is dominated by the pressure of the degenerate relativistic electrons, the pressure and density are related by a polytropic law

$$P = K\rho^{4/3}.$$

[6 marks]

e) Show that the degenerate electrons become relativistic when

$$n_e \gg \left(\frac{m_e c}{h}\right)^3,$$

where m_e is the electron mass.

[6 marks]

Question B4

a) The equation of state for an adiabatic gas can be written as $P = K\rho^{\gamma}$, and hence the adiabatic exponent γ can be defined as

$$\gamma = \left(\frac{\partial \ln P}{\partial \ln \rho}\right)_S ,$$

where subscript S indicates that the partial derivative is taken at constant entropy, i.e. without any heat exchange. Assuming that the equation of state of an ideal gas, $P = \frac{\mathcal{R}}{\mu}\rho T$, also applies show that

$$\left(\frac{\partial \ln T}{\partial \ln P}\right)_S = \frac{\gamma - 1}{\gamma}.$$

[3 marks]

b) Derive the Schwarzschild criterion for the onset of convection in an ideal gas, namely

$$\frac{d \ln T}{d \ln P} > \frac{\gamma - 1}{\gamma},$$

stating clearly any assumptions which are used.

[11 marks]

c) When computing the structure of a star, it is necessary to determine the temperature gradient at each point in the model. Provide a brief explanation of how this is done in practice, and provide arguments to justify on physical grounds the approach that you have outlined.

[4 marks]

d) In a region of convective instability near the surface of a solar-type star of total mass M, the temperature and pressure are related approximately by the expression $P = KT^{5/2}$. Show that the temperature gradient for an ideal gas in hydrostatic equilibrium in this convection zone is given by

$$\frac{dT}{dr} = -\frac{2Gm(r)\mu}{5\mathcal{R}r^2}.$$

Further, assuming that the mass in the convection zone is small compared to M, show that at a depth h measured from the top of the convection zone, the temperature is approximately given by

$$T = T_{\mathsf{S}} + \frac{2GM\mu}{5\mathcal{R}R^2}h,$$

when h is small compared to R and T_{S} is the temperature at the top of the convection zone.

[7 marks]

End of Paper