MATH6502 Example Sheet 3. Hand in all questions from section A.
Cover sheet with DEPARTMENT/TUTOR/YOUR NAME & signed.
Due into Maths room 6.10 by 2pm on Wednesday 29 October.

Section A

1. The Laplacian in two dimensions can be written in two different ways:

$$\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \qquad \nabla^2 f(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2},$$

where r and θ are polar coordinates: $x = r\cos\theta$ and $y = r\sin\theta$. Find the Laplacian of the following functions:

(a) $f(x,y) = 3x^2 + 2xy$ (b) $f(r,\theta) = 3r^2\cos^2\theta + r^2\sin 2\theta$ (c) $f(r,\theta) = Ar^n\cos n\theta + Br^n\sin n\theta$.

2. Find the general solution to each of these ordinary differential equations (in which k is a parameter not a variable):

(a) $\dfrac{\mathrm{d}f}{\mathrm{d}t} + k^2 f = 0$ (b) $\dfrac{\mathrm{d}^2 f}{\mathrm{d}x^2} - k^2 f = 0$ (c) $r^2\dfrac{\mathrm{d}^2 f}{\mathrm{d}r^2} + r\dfrac{\mathrm{d}f}{\mathrm{d}r} - k^2 f = 0,$ (d) $\dfrac{\mathrm{d}^2 f}{\mathrm{d}\theta^2} + k^2 f = 0.$

Hint: use the trial functions $f = e^{\lambda t}$, $f = e^{\lambda x}$, $f = r^m$ and $f = e^{\lambda\theta}$.

Section B

1. Using the extended chain rule (MATH6501):

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}$$
$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

and the polar coordinate definitions $x = r\cos\theta$ and $y = r\sin\theta$, prove that

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

2. Find the solution $f(x,t)$ to the one-dimensional heat equation:

$$\frac{\partial f}{\partial t} = \kappa\frac{\partial^2 f}{\partial x^2}$$

which satisfies:

$$f(0,t) = 0 \qquad f(L,t) = 0 \qquad f(x,0) = x.$$