# UNIVERSITY COLLEGE LONDON
# DEPARTMENT OF PHYSICS AND ASTRONOMY

## 2B21 MATHEMATICAL METHODS IN PHYSICS AND ASTRONOMY

## Problem Sheet M5 (2003–2004)

Solutions to be handed in on Tuesday 13 November 2003

1. By separating variables, show that the solution of the differential equation

$$(1+x^2)\,\frac{dy}{dx} - 2x^3y = 0\,,$$

for which $y = 1$ at $x = 0$ is $y = (1+x^2)^{-1}\,e^{x^2}$. **[4 marks]**

Now solve by the Frobenius method, finding the indicial equation and **[2 marks]**
recurrence relation. **[2 marks]**

By expanding the exact solution, show that the two approaches give the same result up to at least the $x^4$ term. **[6 marks]**

2. Show that the second order differential equation

$$(2x+x^2)\,\frac{d^2y}{dx^2} + (1+x)\,\frac{dy}{dx} - p^2y = 0$$

has two solutions of the form below with $k = 0$ or $k = \frac{1}{2}$:

$$y = \sum_{n=0}^{\infty} a_n\, x^{n+k}\,, \qquad a_0 \neq 0$$ **[6 marks]**

Derive the recurrence relation

$$\frac{a_{n+1}}{a_n} = -\frac{(n+k)^2 - p^2}{(n+k+1)(2n+2k+1)}\,.$$ **[4 marks]**

Use the d'Alembert ratio test to determine the range of values of $x$ for which the series converges. **[3 marks]**

In the special case where $p$ is a positive integer, show that the $k = 0$ series terminates at $n = p$. **[3 marks]**

Denote the resulting polynomial by $T_p(x)$. If $T_p(0) = 1$, show that to order $x$ the polynomials satisfy

$$2\,T_p(x)\,T_q(x) = T_{p+q}(x) + T_{p-q}(x)\,,$$

where $q$ is another positive integer with $p \geq q$. **[4 marks]**