University College London

Department of Physics and Astronomy

2B21 Mathematical Methods in Physics & Astronomy

Suggested Solutions for Problem Sheet M2 (2003–2004)

1. In matrix notation

$$\underline{r}_2 = \underline{A}\,\underline{r}_1 \quad \text{and} \quad \underline{r}_3 = \underline{B}\,\underline{r}_2\,,$$

where

$$\underline{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \underline{B} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \qquad \textbf{[2]}$$

Hence

$$\underline{r}_3 = \underline{B}\,\underline{A}\,\underline{r}_1 = \underline{C}\,\underline{r}_1\,,$$

where

$$\underline{C} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad \textbf{[3]}$$

The new components are therefore

$$x_3 = y_1\,, \quad y_2 = -x_1\,, \quad z_3 = z_1\,. \qquad \textbf{[1]}$$

The resultant rotation is 90° about the z-axis. **[1]**

2. The simultaneous equations are

$$\begin{aligned} 3x_1 - 2x_2 - x_3 &= 4\,, \\ 2x_1 + x_2 + 2x_3 &= 10\,, \\ x_1 + 3x_2 - 4x_3 &= 5\,. \end{aligned}$$

The determinant of the matrix is

$$\Delta = |\underline{A}| = \begin{vmatrix} 3 & -2 & -1 \\ 2 & 1 & 2 \\ 1 & 3 & -4 \end{vmatrix} = 3(-4-6) + 2(-8-2) - 1(6-1) = -55\,. \qquad \textbf{[2]}$$

By Cramer's rule,

$$\Delta \times x_1 = \begin{vmatrix} 4 & -2 & -1 \\ 10 & 1 & 2 \\ 5 & 3 & -4 \end{vmatrix} = 4(-4-6) + 2(-4--10) - (30-5) = -40-100-25 = -165.$$

Hence $x_1 = 3$. **[2]**

$$\Delta \times x_2 = \begin{vmatrix} 3 & 4 & -1 \\ 2 & 10 & 2 \\ 1 & 5 & -4 \end{vmatrix} = 3(-40-10)-4(-8-2)-(10-10) = -150+40 = -110.$$

Hence $x_2 = 2$. **[2]**

$$\Delta \times x_3 = \begin{vmatrix} 3 & -2 & 4 \\ 2 & 1 & 10 \\ 1 & 3 & 5 \end{vmatrix} = 3(5-30)+2(10-10)+4(6-1) = -55\,.$$

Hence $x_1 = 1$. **[2]**

3. The determinant of the matrix is

$$|\underline{A}| = \begin{vmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \\ 3 & 3 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 0 & -2 & 1 \\ 0 & -6 & -1 \end{vmatrix} = 8\,.$$ **[2]**

The cofactor matrix is

$$\underline{C} = \begin{pmatrix} -4 & 4 & 0 \\ -3 & -1 & 6 \\ 5 & -1 & -2 \end{pmatrix}.$$ **[3]**

The adjoint matrix is merely the transpose of $\underline{C}$;

$$\underline{A}^{\rm adj} = \underline{C}^T = \begin{pmatrix} -4 & -3 & 5 \\ 4 & -1 & -1 \\ 0 & 6 & -2 \end{pmatrix}.$$ **[1]**

Hence the inverse matrix

$$\underline{A}^{-1} = \frac{1}{|\underline{A}|}\,\underline{A}^{\rm adj} = \frac{1}{8}\begin{pmatrix} -4 & -3 & 5 \\ 4 & -1 & -1 \\ 0 & 6 & -2 \end{pmatrix}.$$ **[1]**

Thus

$$\underline{B} = \underline{A}^{-1}\underline{D} = \frac{1}{8}\begin{pmatrix} -4 & -3 & 5 \\ 4 & -1 & -1 \\ 0 & 6 & -2 \end{pmatrix}\begin{pmatrix} 10 & -6 & 6 \\ 9 & -6 & 5 \\ 15 & -10 & 11 \end{pmatrix}$$

$$= \begin{pmatrix} (-40-27+75) & (24+18-50) & (-24-15+55) \\ (40-9-15) & (-24+6+10) & (24-5-11) \\ 0+54-30) & (0-36+20) & (0+30-22) \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \\ 3 & -2 & 1 \end{pmatrix}.$$

[3]