1. State the divergence theorem. **[2 marks]**

Calculate the integral of the divergence of the vector field

$$\underline{F} = xy\,\underline{\hat{e}}_x + yz^2\,\underline{\hat{e}}_y + \underline{\hat{e}}_z$$

over the volume of the hemisphere defined by $x^2 + y^2 + z^2 \leq 16$ and $z \geq 0$. **[8 marks]**

Write down the z and the radial components of $\underline{F}$ and use them to calculate explicitly the flux of $\underline{F}$ through the base and the curved surface of the hemisphere. Hence verify the divergence theorem in this case. **[10 marks]**

Note that in spherical polar coordinates

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

and the radius vector points in the direction

$$\hat{r} = \sin\theta\cos\phi\,\underline{\hat{e}}_x + \sin\theta\sin\phi\,\underline{\hat{e}}_y + \cos\theta\,\underline{\hat{e}}_z\ .$$

The element of area perpendicular to the radius vector is

$$dA_r = r^2\,\sin\theta\,d\theta\,d\phi\ ,$$

and the corresponding element of volume is

$$dV = r^2\,dr\,\sin\theta\,d\theta\,d\phi\ .$$

Integrals of the type $\int \sin^{2n+1}\theta\,\cos^{2m}\theta\,d\theta$ (m and n integers) can be evaluated by using the substitution $t = \cos\theta$, $dt = -\sin\theta\,d\theta$.

SOLUTION

The divergence theorem states that

$$\int_V \underline{\nabla} \cdot \underline{F}\, dV = \int_S \underline{F} \cdot \hat{n}\, dS\ ,$$

where S is the closed surface surrounding the volume V and $\hat{n}$ is a unit vector directed along the outward normal to S. **[2]**

The divergence of the vector field $\underline{F} = xy\,\hat{\underline{e}}_x + yz^2\,\hat{\underline{e}}_y + \hat{\underline{e}}_z$ is

$$\underline{\nabla} \cdot \underline{F} = y + z^2\ .$$ **[2]**

The volume element in spherical polar coordinates is $dV = r^2\, dr\, \sin\theta\, d\theta\, d\phi$, so that

$$I = \int_V \underline{\nabla} \cdot \underline{F}\, dV = \int_0^4 r^2\, dr \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{2\pi} d\phi\, (r \sin\theta\, \sin\phi + r^2 \cos^2\theta)\ .$$ **[2]**

Note that the radius is at $r = 4$ and the <u>hemi</u>sphere condition has been introduced by integrating $0 \le \theta \le \frac{1}{2}\pi$.

Now the first term in the bracket is killed by the integration over ϕ. Hence

$$I = 2\pi \int_0^4 r^4\, dr \int_0^{\pi/2} \sin\theta\, \cos^2\theta\, d\theta = -2\pi \int_0^4 r^4\, dr \int_{\theta=0}^{\theta=\pi/2} \cos^2\theta\, d(\cos\theta)$$

$$= \frac{2\pi}{3} \int_0^4 r^4\, dr = \frac{2\pi}{15}\, 4^5 = \frac{2048\pi}{15}\ .$$ **[4]**

Evaluating now the flux through the flat surface at $z = 0$, only the z-component of $\underline{F}$ contributes. Now $F_z = 1$ and $\hat{n} = -\hat{\underline{e}}_z$, which means that

$$J_z = -\int_0^4 r\, dr \int_0^{2\pi} d\phi\, F_z = -\int_0^4 r\, dr \int_0^{2\pi} d\phi = -2\pi \int_0^4 r\, dr = -16\pi\ .$$ **[2]**

On the curved surface we want the radial component of the flux;

$$F_r = \underline{F} \cdot \hat{r} = \underline{F} \cdot \underline{r}/r$$

$$= F_x \sin\theta \cos\phi + F_y \sin\theta \sin\phi + F_z \cos\theta = xy \sin\theta \cos\phi + yz^2 \sin\theta \sin\phi + \cos\theta$$

$$= 16 \sin^3\theta \sin\phi \cos^2\phi + 64 \sin^2\theta \cos^2\theta \sin^2\phi + \cos\theta\ .$$ **[2]**

The radial flux is

$$J_r = 16 \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} d\phi \, [16 \sin^3\theta \sin\phi \cos^2\phi + 64 \sin^2\theta \cos^2\theta \sin^2\phi + \cos\theta].$$

Taking the terms one-by-one,

$$J_r^{(1)} = 256 \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \sin^3\theta \sin\phi \cos^2\phi = 0 \qquad [\mathbf{1}]$$

because the integrand is odd in ϕ.

$$J_r^{(2)} = 1024 \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \sin^2\theta \cos^2\theta \sin^2\phi = 1024\pi \int_0^{\pi/2} \sin^3\theta \cos^2\theta \, d\theta$$

$$= -1024\pi \int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d(\cos\theta) = -1024\pi \int_{\theta=0}^{\theta=\pi/2} \cos^2\theta (1 - \cos^2\theta) \, d(\cos\theta)$$

$$= -1024\pi \left[\frac{1}{3}\cos^3\theta - \frac{1}{5}\cos^2\theta \right]_0^{\pi/2} = \frac{2048\pi}{15} \, . \qquad [\mathbf{2}]$$

$$J_r^{(3)} = 16 \int_0^{\pi/2} \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \cos\theta = 32\pi \int_0^{\pi/2} \sin\theta \, \cos\theta \, d\theta$$

$$= 16\pi \int_0^{\pi/2} \sin 2\theta \, d\theta = -8\pi \left[\cos 2\theta \right]_0^{\pi/2} = 16\pi \, . \qquad [\mathbf{2}]$$

Adding all the contributions together,

$$J_z + J_r^{(1)} + J_r^{(2)} + J_r^{(3)} = -16\pi + 0 + \frac{2048\pi}{15} + 16\pi = \frac{2048\pi}{15} \, ,$$

which verifies the divergence theorem in this case. [**1**]

First two marks are bookwork.