MATHEMATICAL METHODS IN PHYSICS & ASTRONOMY

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Colin Wilkin

(cw@hep.ucl.ac.uk)

The first year 1B21 course and the second year 2B21 course should together provide all the necessary mathematical techniques for the understanding of the compulsory (core) courses in the whole Physics and Astronomy and related degree programmes. The two second year courses which rely heavily on the mathematical formalism developed in 2B21 are Quantum Mechanics and Electromagnetic theory. Since Electromagnetism is in the second semester, the course will start with the things needed for Quantum Mechanics, *viz* matrices and eigenvalues, partial differential equations, and the solutions of the Legendre differential equation. By imposing boundary conditions on this, we will derive quantisation of angular momentum about the same time as you will see it in the Quantum Physics course. The properties of the Legendre polynomials and spherical harmonics derived here should help you understand some of the Quantum phenomena. Though you will have met the gradient differential operator in the 1B21 course, Electromagnetism requires that you be able to apply also the divergence and curl operators, as well as use the divergence and Stokes' theorem which are introduced at the end of the course. The Fourier analysis is also of use in second year Electromagnetism, but it will required in many other courses such as modern optics or signal processing.

Even if it is not true, the assumption has to be made that you have understood and can reproduce most of the first year 1B21 material, including the Notes on Integration. You will have to remember the contents of the Formulae Sheet because it will not be issued in second year examinations. It can therefore do no harm at all if you spent some time going through last year's notes on the subject.

There are about 34 lectures in total in the 2B21 course. The methodology is rather similar to 1B21 but, because there are no associated problem solving classes, there are about 9 discussion classes where I will go over examples. Following suggestions from students in past years, most of these examples will be chosen by me so that they fit well with the course and the problem sheets. That seemed to work quite well in the past, but it is not intended to stop students raising their own questions. The discussion classes will normally be in the second slot on Thursday afternoons, except for the first week when I shall lecture for two hours instead — it is hard to discuss problems before getting well into the course.

The written summer examination counts for 90% of the assessment. Following advice from External Examiners, the rubric was changed last year such that you will only be allowed to attempt **FIVE** questions. This should be borne in mind if you look at old examination papers. As for the 1B21 course, there is an examination just before Christmas. In order to allow for assimilation of material, this will only be two hours long and not cover the very end of the course. You will then be asked to attempt four out of the six questions. Though it counts 2% to the assessment, the mid-sessional examination is primarily diagnostic; it is a major element in the decision of whether you should take the B8 option offered by the Mathematics Department in the second semester. As you can see from past papers, the 2B21 examination tends to have a little more bookwork than the 1B21.

The remaining 8% of the continuous assessment is provided by the best 8 out of the 10 weekly problem sheets, that are handed out as a package at the start of the course. These are due in on Tuesdays (except for the last one which is due in after the Christmas vacation) and will be subject to the standard Departmental penalty for late handing-in. Since I want to put the solutions onto the WEB on Thursdays, this provides a natural deadline for all except students with valid medical excuses. Most, if not all, of the questions on last year's summer examination will be found on this year's problem sheets. This effectively means that you will eventually accumulate all of last year's examination solutions. Please note that, to satisfy College regulations, 35% of course work must be submitted in order that a course be deemed to be complete. **No credit is given towards your degree for incomplete courses.**

The recommended book from the first year 1B21 course, *Mathematical Methods in the Physical Sciences*, by Mary L. Boas (Wiley), covers all the material in the present half-unit. A more specialised second year book is *Mathematical Methods for Physics and Engineering*, by K.F. Riley, M.P. Hobson and S.J. Bence (Cambridge University Press). *Advanced Engineering Mathematics* by E. Kreyszig (Wiley) is good, though it has a little less on Legendre functions than you will need. Those students who are keen on mathematics should appreciate the more challenging presentation in *Mathematical Methods for Physicists*, by G.B. Arfken and H.-J. Weber (Academic Press). The *Further Engineering Mathematics* by K.A. Stroud provides some useful examples, but it does not cover the whole of the course, having very little on the Legendre functions.

There is a timetable clash for students on the Medical Physics course on two afternoons per term. Once the times have been finalised, I shall contact these students and arrange a tutorial to go over the missing material.

The danger in any second year Mathematics course is that some students get bored and frustrated because they are getting lost. The natural temptation is that they then sit at the back of the theatre and chat with their neighbours. This will not help them with the understanding of the course and it is terribly disrupting to other students and sometimes puts the lecturer out of his stride. Please refrain or I shall have to shout at you to get quiet. If you do read a newspaper, please read a good quality one. The Sun, Mirror, or Metro are (in general) not going to improve your eventual job prospects — look at the job adverts in the latter!

Apart from the syllabus, the rest of the hand-out consists of a summary of integration volumes and vector operators in cylindrical and spherical polar coordinates, together with a useful list of identities involving vector operators.

Provisional Syllabus 2003/2004

Linear Vector Spaces and Matrices

Definition and properties of determinants, especially 3×3. **[1.5]**
Revision of real 3-dimensional vectors, Complex linear vector spaces, Linear transformations and their representation in terms of matrices, Multiple transformations and matrix multiplication. **[3]**
Properties of matrices, Special matrices, Matrix inversion, Solution of linear simultaneous equations. **[4]**
Eigenvalues and eigenvectors, Eigenvalues of unitary and Hermitian matrices, Real quadratic forms, Normal modes of oscillation. **[4]**

Partial Differential Equations

Superposition principle for linear homogenious partial differential equations, Separation of variables in Cartesian coordinates, Boundary conditions, One-dimensional wave equation, Derivation of Laplace's equation in spherical polar coordinates, Separation of variables in spherical polar coordinates, the Legendre differential equation, Solutions of degree zero. **[3.5]**

Series Solution of Ordinary Differential Equations

Derivation of the Frobenius method, Application to linear first order equations, Singular points and convergence, Application to second order equations. **[2]**

Legendre Functions

Application of the Frobenius method to the Legendre equation, Range of convergence, Quantisation of the ℓ index, Generating function for Legendre polynomials, Recurrence relations, Orthogonality of Legendre functions, Expansion in series of Legendre polynomials, Solution of Laplace's equation for a conducting sphere, Associated Legendre functions, Spherical harmonics. **[4]**

Fourier Analysis

Fourier series, Periodic functions, Derivation of basic formulae, Simple applications, Gibbs phenomenon (empirical), Differentiation and integration of Fourier series, Parseval's identity, Complex Fourier series. **[2.5]**
Fourier transforms, Derivation of basic formulae and simple application, Dirac delta function, Convolution theorem. **[2.5]**

Vector Operators

Gradient, divergence, curl and Laplacian operators in Cartesian coordinates, Flux of a vector field, Divergence theorem, Stokes' theorem, Coordinate-independent definitions of vector operators. Derivation of vector operators in spherical and cylindrical polar coordinates. **[6]**

Some useful three-dimensional vector identities

Assume that ϕ and ψ are differentiable scalar fields and that $\underline{A}$ and $\underline{B}$ are differentiable vector fields. Then the following results are true in general:

$$\begin{aligned}
\textit{div grad}\ \phi &= \nabla^{\mathbf{2}}\,\phi \\
\textit{curl grad}\ \phi &= \nabla\ \times\ \nabla\,\phi = \mathbf{0} \\
\textit{div curl}\ \underline{A} &= \nabla\cdot(\nabla\times\underline{\mathbf{A}}) = \mathbf{0} \\
\textit{curl curl}\ \underline{A} &= \nabla\times(\nabla\times\underline{\mathbf{A}}) = \nabla(\nabla\cdot\underline{\mathbf{A}}) - \nabla^{\mathbf{2}}\underline{\mathbf{A}} \\
\textit{grad div}\ \underline{A} &= \nabla(\nabla\cdot\underline{\mathbf{A}}) = \mathbf{0} \\
\nabla\cdot(\phi\,\underline{\mathbf{A}}) &= \phi(\nabla\cdot\underline{\mathbf{A}}) + \underline{\mathbf{A}}\cdot(\nabla\,\phi) \\
\nabla\times(\phi\,\underline{\mathbf{A}}) &= \phi(\nabla\times\underline{\mathbf{A}}) - \underline{\mathbf{A}}\times(\nabla\,\phi) \\
\nabla\cdot(\underline{\mathbf{A}}\times\underline{\mathbf{B}}) &= \underline{B}\cdot(\nabla\times\underline{\mathbf{A}}) - \underline{\mathbf{A}}\cdot(\nabla\times\underline{\mathbf{B}}) \\
\nabla\times(\underline{\mathbf{A}}\times\underline{\mathbf{B}}) &= (\underline{B}\cdot\nabla)\underline{\mathbf{A}} - (\underline{\mathbf{A}}\cdot\nabla)\underline{\mathbf{B}} - \underline{\mathbf{B}}(\nabla\cdot\underline{\mathbf{A}}) + \underline{\mathbf{A}}(\nabla\cdot\underline{\mathbf{B}}) \\
\nabla(\underline{\mathbf{A}}\cdot\underline{\mathbf{B}}) &= \underline{B}\times(\nabla\times\underline{\mathbf{A}}) + (\underline{\mathbf{B}}\cdot\nabla)\underline{\mathbf{A}} + \underline{\mathbf{A}}\times(\nabla\times\underline{\mathbf{B}}) + (\underline{\mathbf{A}}\cdot\nabla)\underline{\mathbf{B}} \\
\nabla\cdot(\nabla\phi\times\nabla\psi) &= 0
\end{aligned}$$

Vector Operators in Spherical Polar Coordinates

Coordinates:

$$\begin{aligned} x = &\ r\sin\theta\cos\phi\ , \\ y = &\ r\sin\theta\sin\phi\ , \\ z = &\ r\cos\theta\ . \end{aligned}$$

Basis vectors:

$$\begin{aligned} \underline{\hat{e}}_r = &\ \sin\theta\cos\phi\,\underline{\hat{e}}_x + \sin\theta\sin\phi\,\underline{\hat{e}}_y + \cos\theta\,\underline{\hat{e}}_z\ , \\ \underline{\hat{e}}_\theta = &\ \cos\theta\cos\phi\,\underline{\hat{e}}_x + \cos\theta\sin\phi\,\underline{\hat{e}}_y - \sin\theta\,\underline{\hat{e}}_z\ , \\ \underline{\hat{e}}_\phi = &\ -\sin\phi\,\underline{\hat{e}}_x + \cos\phi\,\underline{\hat{e}}_y\ , \end{aligned}$$

Volume element

$$d^3r = r^2\ \sin\theta\,dr\,d\theta\,d\phi\ .$$

Gradient:

$$\nabla\,\mathbf{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{r}}\,\underline{\hat{\mathbf{e}}}_{\mathbf{r}} + \frac{\mathbf{1}}{\mathbf{r}}\,\frac{\partial \mathbf{f}}{\partial \theta}\,\underline{\hat{\mathbf{e}}}_\theta + \frac{\mathbf{1}}{\mathbf{r}\sin\theta}\,\frac{\partial \mathbf{f}}{\partial \phi}\,\underline{\hat{\mathbf{e}}}_\phi\ .$$

Divergence:

$$\nabla\cdot\underline{\mathbf{F}} = \frac{\mathbf{1}}{\mathbf{r^2}}\,\frac{\partial}{\partial \mathbf{r}}\left(\mathbf{r^2}\,\mathbf{F_r}(\mathbf{r},\theta,\phi)\right) + \frac{\mathbf{1}}{\mathbf{r}\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\mathbf{F}_\theta(\mathbf{r},\theta,\phi)\right) + \frac{\mathbf{1}}{\mathbf{r}\sin\theta}\,\frac{\partial}{\partial\phi}\left(\mathbf{F}_\phi(\mathbf{r},\theta,\phi)\right).$$

Laplacian:

$$\nabla^2 V = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial}{\partial\phi}\left(\frac{\partial V}{\partial\phi}\right)\ .$$

Curl:

$$\begin{aligned} (\nabla\times\underline{\mathbf{F}})_r &= \frac{1}{r\ \sin\theta}\left\{\frac{\partial}{\partial\theta}\{\sin\theta\,F_\phi(r,\theta,\phi)\} - \frac{\partial}{\partial\phi}\,F_\theta(r,\theta,\phi)\right\}\ . \\ (\nabla\times\underline{\mathbf{F}})_\theta &= \frac{1}{r\ \sin\theta}\left\{\frac{\partial}{\partial\phi}\,F_r(r,\theta,\phi) - \sin\theta\,\frac{\partial}{\partial r}\ \{r\,F_\phi(r,\theta,\phi)\}\right\}\ , \\ (\nabla\times\underline{\mathbf{F}})_\phi &= \frac{1}{r}\left\{\frac{\partial}{\partial r}\ \{r\,F_\theta(r,\theta,\phi)\} - \frac{\partial}{\partial\theta}\,F_r(r,\theta,\phi)\right\}\ . \end{aligned}$$

Vector Operators in Cylindrical Polar Coordinates

Coordinates:

$$\begin{aligned} x = & \quad r\cos\theta\,, \\ y = & \quad r\sin\theta\,, \\ z = & \quad\quad z\,. \end{aligned}$$

Basis vectors:

$$\begin{aligned} \underline{\hat{e}}_r &= \cos\theta\,\underline{\hat{e}}_x + \sin\theta\,\underline{\hat{e}}_y\,, \\ \underline{\hat{e}}_\theta &= -\sin\theta\,\underline{\hat{e}}_x + \cos\theta\,\underline{\hat{e}}_y\,, \\ \underline{\hat{e}}_z &= \underline{\hat{e}}_z\,. \end{aligned}$$

Volume element

$$d^3r = r\,dr\,d\theta\,dz\,.$$

Gradient:

$$\nabla\,\mathbf{f} = \frac{\partial \mathbf{f}}{\partial \mathbf{r}}\,\underline{\hat{\mathbf{e}}}_{\mathbf{r}} + \frac{\mathbf{1}}{\mathbf{r}}\,\frac{\partial \mathbf{f}}{\partial \theta}\,\underline{\hat{\mathbf{e}}}_\theta + \frac{\partial \mathbf{f}}{\partial \mathbf{z}}\,\underline{\hat{\mathbf{e}}}_{\mathbf{z}}\,.$$

Divergence:

$$\nabla\cdot\underline{\mathbf{F}} = \frac{\mathbf{1}}{\mathbf{r}}\left\{\frac{\partial}{\partial \mathbf{r}}\left(\mathbf{r}\,\mathbf{F}_{\mathbf{r}}(\mathbf{r},\theta,\mathbf{z})\right) + \frac{\partial}{\partial\theta}\mathbf{F}_\theta(\mathbf{r},\theta,\mathbf{z}) + \mathbf{r}\,\frac{\partial}{\partial \mathbf{z}}\mathbf{F}_{\mathbf{z}}(\mathbf{r},\theta,\mathbf{z})\right\}\,.$$

Laplacian:

$$\nabla^2 V = \frac{1}{r}\,\frac{\partial}{\partial r}\left(r\,\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\,\frac{\partial^2 V}{\partial\theta^2} + \frac{\partial^2 V}{\partial z^2}\,.$$

Curl:

$$(\nabla\times\underline{\mathbf{F}})_r = \frac{1}{r}\,\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\,.$$

$$(\nabla\times\underline{\mathbf{F}})_\theta = \frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\,.$$

$$(\nabla\times\underline{\mathbf{F}})_z = \frac{1}{r}\,\left\{\frac{\partial}{\partial r}\,\{r\,F_\theta(r,\theta,\phi)\} - \frac{\partial}{\partial\theta}\,F_z(r,\theta,\phi)\right\}\,.$$