UNIVERSITY OF LONDON
(University College London)

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PHYSICS B221 : MATHEMATICAL METHODS IN PHYSICS

Credit will be given for all work done. [For guidance, a student should aim to answer correctly the equivalent of FOUR complete questions in the time available].

1. If ϕ is a scalar point function and **A** is a vector point function, give expressions for

   $\nabla\phi$, $\nabla.\mathbf{A}$, $\nabla\times\mathbf{A}$, $\nabla^2\phi$ in Cartesian coordinates. [4]

   Verify the identities

   $$\nabla.(\phi\ \mathbf{A}) = (\nabla\phi\ ).\mathbf{A} + \ \phi(\nabla.\mathbf{A})$$

   $$\nabla.(\mathbf{A}\times\mathbf{B}) = \mathbf{B}.(\nabla\times\mathbf{A}) - \mathbf{A}.(\nabla\times\mathbf{B})$$

   $$\nabla\times\nabla\phi = 0$$

   $$\nabla.(\nabla\times\mathbf{A}) = 0$$ [8]

   Write down expressions defining a *solenoidal field* and a *conservative field*. [4]

   In a vacuum, one of Maxwell's equations of electromagnetism may be written $\nabla.\mathbf{B} = 0$, where **B** is the magnetic induction. Show that **B** can be written as the curl of the vector potential **A**. What type of field is **B**? [4]

2. By examining the way in which a scalar field changes from one level surface to another, write down in Cartesian coordinates, the definition of the gradient of a scalar field. [4]

   Write down the transformations between Cartesian coordinates (x, y, z) and spherical polar coordinates (r, θ, ϕ), and the expression for the gradient of a scalar field in spherical polar coordinates. [4]

   Evaluate ∇r^n in both Cartesian and spherical polar coordinates. [2]

   Verify Stoke's theorem

   $$\int_S \nabla\times\mathbf{A}.\hat{\mathbf{n}}\ \ dS = \int_\gamma \mathbf{A}.\mathbf{d}\ell$$

   by direct calculation for the vector field $\mathbf{A} = (x-2y)\ \hat{\mathbf{i}} - 2yz^2\hat{\mathbf{j}} - 2y^2z\hat{\mathbf{k}}$ for the upper half surface of the sphere $x^2 + y^2 + z^2 = 4$ and its boundary with the (x, y) plane. [10]

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3. Write down definitions of

   i) the transpose of a matrix
   ii) the adjoint of a matrix
   iii) a Hermitian matrix
   iv) a Unitary matrix [4]

   Show that if two matrices **A** and **B** are Hermitian, then $(\mathbf{AB})^\dagger = \mathbf{B}^\dagger\mathbf{A}^\dagger$. [3]

   Show that the eigenvalues of a Hermitian matrix are real, and the corresponding eigenvectors of distinct eigenvalues are orthogonal. [6]

   Show that the eigenvalues of a Unitary matrix have magnitude unity. [3]

   If **C** is a matrix, which is not necessarily Hermitian, show that it is always possible to write **C** as the sum of two Hermitian matrices in the form $\mathbf{C} = \mathbf{A} + \mathrm{i}\mathbf{B}$. [4]

4. A guitar string of length l is fixed at its ends, $x=0$ and $x=l$, and has a displacement $y(x,t)$ perpendicular to its equilibrium position which satisfies

$$\frac{\partial^2 y(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2} = 0$$

   where c is a real constant. Use the method of separation of variables to show that if at time $t=0$, the string is at rest, then

$$y(x,t) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi ct}{l}\right) \qquad [8]$$

   If the displacement at time $t=0$ is of the form

$$y(x,0) = \begin{cases} \alpha x, & 0 \le x \le l/2 \\ \alpha(l-x), & l/2 \le x \le l \end{cases}$$

   show that the subsequent displacement is given by

$$y(x,t) = \frac{4l\alpha}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\sin\frac{(2m+1)\pi x}{l}\cos\frac{(2m+1)\pi ct}{l} \qquad [12]$$

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5. Laguerre's differential equation is given by

$$xy'' + (1-x)y' + py = 0$$

By writing

$$y(x) = \sum_{\lambda=0}^{\infty} a_\lambda x^{k+\lambda}$$

show that *k=0* is a solution of the *indicial equation*. [4]

In this case, develop a recursion relation between successive coefficients, a_j and write down a general expression for each coefficient in the series. [6]

Thus, when *k=0*, write down the solution for *y(x)* and show that when *p=n*, where *n* is a positive integer, the series for *y(x)* terminates and for each *n* , a solution, $y = L_n(x)$ can be found. [6]

Write down the solutions for *n=0, 1, 2, 3.* [4]

6. A function, *f(x)* which is periodic over the interval *(-L, +L)*, may be represented by the Fourier series

$$f(x) = \frac{1}{2L}\int_{-L}^{+L} f(t)\ dt \ + \ \frac{1}{L}\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \ + \ \frac{1}{L}\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

where

$$a_n = \int_{-L}^{+L} f(t)\cos\frac{n\pi t}{L}\ dt$$

$$b_n = \int_{-L}^{+L} f(t)\sin\frac{n\pi t}{L}\ dt$$

In the limit that the period *L* tends to infinity, and the periodic function is transformed to a pulse, show that we can write

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$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\omega) \exp(-i\omega x) \; d\omega$$

where

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) \exp(i\omega t) \; dt$$

is the Fourier Transform of *f(x)*. [8]

A function *f(t)* is defined such that

$$f(t) = \sin \omega_0 t \qquad |t| \le N\pi / \omega_0$$

$$f(t) = 0 \qquad |t| \ge N\pi / \omega_0$$

where *N* is a positive integer. Sketch the function, explaining whether it is even or odd. [2]

Calculate the Fourier transform, $g(\omega)$ of *f(t)*, again sketching your result. [6]

For large ω_0 and $\omega \approx \omega_0$, determine the zeroes of $g(\omega)$. Give a physical example to which this Fourier Transform might correspond. [4]

7. A generating function for the Legendre polynomials, $P_l(x)$, can be written

$$g(x,t) = \left(1 - 2xt + t^2\right)^{-1/2} = \sum_{l=0}^{\infty} P_l(x) \; t^l$$

where $|t| < 1$ and *l* is an integer. Show that $P_l(1) = 1$ for all *l*. [6]

By differentiating the recurrence relation, in the one case with respect to *t* and in the other case with respect to *x*, show that

$$(2n+1) x P_n(x) = (n+1) P_{n+1}(x) + n P_{n-1}(x)$$
$$P'_{n+1}(x) + P'_{n-1}(x) = 2x P'_n(x) + P_n(x)$$

[8]

Show from the generating function that $P_0(x) = 1$, and hence, using the first recurrence relation above, deduce the next three Legendre polynomials [6]

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