UNIVERSITY OF LONDON
(University College London)

BSc Degree 1999

PHYS2B21

MATHEMATICAL METHODS IN PHYSICS AND ASTRONOMY

QUESTIONS

Credit will be given for all correct work done. [For guidance: a student should aim to answer correctly the equivalent of four complete questions in the time available.]

The numbers in square brackets at the right-hand edge of the paper indicate the provisional allocation of maximum marks for each section of a question.

1. **(a)** Show that there is a value of the constant *a* for which the vector

$$\mathbf{A} = (axy - z^3)\hat{\mathbf{i}} + (a-2)x^2\hat{\mathbf{j}} + (1-a)xz^2\hat{\mathbf{k}}$$

is conservative?

[3 marks]

(b) Use Stokes' theorem to relate the line integral

$$\oint_C \mathbf{A}\cdot d\mathbf{r}$$

of the vector

$$\mathbf{A} = 2y\hat{\mathbf{i}} + 3x\hat{\mathbf{j}} - z^2\hat{\mathbf{k}}$$

around the contour C to a surface integral. If C is the closed curve which bounds the surface S of the hemisphere

$$x^2 + y^2 + z^2 = 9 \;;\; z \geq 0$$

use spherical polar co-ordinates to evaluate the surface integral directly.

[7 marks]

(c) Use the identity

$$(\mathbf{a}\wedge\mathbf{b})\wedge(\mathbf{c}\wedge\mathbf{d}) = \mathbf{b}[\mathbf{a}\cdot(\mathbf{c}\wedge\mathbf{d})] - \mathbf{a}[\mathbf{b}\cdot(\mathbf{c}\wedge\mathbf{d})]$$

to show that

$$\left[(\mathbf{A}\wedge\mathbf{B})\wedge(\mathbf{B}\wedge\mathbf{C})\right]\wedge(\mathbf{C}\wedge\mathbf{A}) = \sin\theta\cos\theta\,\mathbf{C}$$

where **A** and **B** are unit vectors, such that $\mathbf{A}\wedge\mathbf{B} = \sin\theta$, and **C** is a unit vector in the direction of $\mathbf{A}\wedge\mathbf{B}$.

[10 marks]

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2. **(a)** Find the directional derivative of the scalar field

$$\phi = 2xy - z^2$$

at the point (2, –1, 1) in the direction towards the point (3, 1, –1).

[5 marks]

(b) Use the divergence theorem to evaluate the integral

$$\oiint_S (\mathbf{F}\cdot\mathbf{n})\, dS$$

where

$$\mathbf{F} = 4xz\hat{\mathbf{i}} - y^2\hat{\mathbf{j}} + yz\hat{\mathbf{k}}$$

n is a unit vector normal to the surface element *dS*, and *S* is the surface of the cube bounded by $x = 0, 1$; $y = 0, 1$; $z = 0, 1$.

[6 marks]

(c) Verify Green's theorem in the plane

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \oint_C \left(P dx + Q dy \right)$$

for the vector

$$\mathbf{F} = (xy + y^2)\hat{\mathbf{i}} + x^2\hat{\mathbf{j}}$$

taken around the closed curve of the region bounded by $y = x$ and $y = x^2$. A is the area enclosed by the closed curve *C*, and the line integral is taken in a counter-clockwise direction,

[9 marks]

3. **(a)** Use Cramer's method to solve, ***for y only***, the simultaneous linear equations

$$\left.\begin{aligned} 2x - 3y + 2z &= 5 \\ x + y - 2z &= 2 \\ -3x + 2y - 3z &= 3 \end{aligned}\right\}$$

[6 marks]

(b) If **A** and **B** are square matrices of the same order,

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$$

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Use this result to prove that for any orthogonal matrix **O**,

$$\det(\mathbf{O}) = \pm 1$$

Prove that a real unitary matrix is a orthogonal and a real hermitian matrix is symmetric.

[8 marks]

(c) The equations

$$\begin{cases} x' = \tfrac{1}{2}\left(x + \sqrt{3}y\right) \\ y' = \tfrac{1}{2}\left(-\sqrt{3}x + y\right) \end{cases} \qquad \begin{cases} x'' = \tfrac{1}{2}\left(-x' + \sqrt{3}y'\right) \\ y'' = -\tfrac{1}{2}\left(\sqrt{3}x' + y'\right) \end{cases}$$

represent two successive rotations of axes in two dimensions. Use matrix methods to find the combined geometrical result of the two rotations.

[6 marks]

4. Explain what is meant by the terms ***regular point***, ***singular point***, ***regular singular point*** and ***essential singularity*** as applied to a linear second order differential equation. What is their relation to the possibility of finding series solutions of the equation?

[7 marks]

Verify that $x = 0$ is a regular singular point of the equation

$$2x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

and show that ***one*** solution is of the form

$$y(x) = Ax^{3/2}\left[1 - \tfrac{2}{5}x + \tfrac{2}{35}x^2 - .....\right]$$

[13 marks]

5. The quadratic form

$$x^2 + y^2 - 3z^2 + 2xy + 6xz - 6yz$$

may be written in the matrix form $\mathbf{X}^T\mathbf{M}\mathbf{X}$, where

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathbf{M} = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{pmatrix}$$

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Show that the eigenvalues of **M** are $\lambda_1 = 2,\ \lambda_2 = 3,\ \lambda_3 = -6$ and find the corresponding normalised eigenvectors. Hence find the orthogonal matrix **S** which diagonalises **M**, ie

$$\mathbf{S}^T\mathbf{M}\mathbf{S} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

[12 marks]

If the quadratic form is written

$$\alpha x'^2 + \beta y'^2 + \gamma z'^2$$

show that $\alpha = \lambda_1$, $\beta = \lambda_2$, $\gamma = \lambda_3$, and find the relation between the primed and unprimed variables.

[8 marks]

6. The function

$$f(x) = (x-1)^2$$

is defined in the interval [0, 2]. Sketch a continuation to the interval [–2, 0] such that it can be represented by a Fourier ***cosine*** series and show that the series may be written in the form

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(n\pi x)}{n^2}$$

[12 marks]

State Parseval's theorem and apply it to the above result to find the value of

$$\sum_{n=1}^{\infty}\frac{1}{n^4}$$

You may assume the integral

$$\int (x-1)^2\cos(ax)\,dx = \frac{(x-1)^2}{a}\sin(ax) + \frac{2(x-1)}{a^2}\cos(ax) - \frac{2}{a^3}\sin(ax)$$

[8 marks]

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7. A two-dimensional rectangular plate is of width $x = w$ and infinitely long in the y direction. The long sides are kept at a temperature of zero except along the bottom edge where the temperature is given by $T(x,0) = f(x) = x$ (This is strictly inconsistent at the point $x = w$, $y = 0$, but you may ignore this). Solve the Laplace equation

$$\nabla^2 T(x,y) = 0$$

by the method of separation of variables, taking the separation constant in the x variable to be $-k^2$, where k is a positive real constant, and show that the solution is

$$T(x,y) = \frac{2w}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-n\pi y/w} \sin\left(\frac{n\pi x}{w}\right)$$

[20 marks]

Useful integrals are:

$$\int x \sin x \, dx = \sin x - x \cos x$$

and

$$\frac{1}{2L} \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & (m \neq n) \\ \frac{1}{2} & (m = n \neq 0) \end{cases}$$

END OF PAPER