UNIVERSITY OF LONDON
(University College London)

Summer Examination.

PHYSICS 2B21: MATHEMATICAL METHODS IN PHYSICS

May 1998. 2.5 HOURS.

Credit will be given for all work done.

[For guidance: A student should aim to answer correctly the equivalent of **FOUR** complete questions in the time available].

The numbers in the square brackets in the right-hand margin indicate the provisional allocation of marks per subsection of a question.

1. If ψ is a scalar point function and $\mathbf{A}$ is a vector point function, give the expressions for (i) $\nabla\psi$, (ii) $\nabla \cdot \mathbf{A}$ and (iii) $\nabla \times \mathbf{A}$ in Cartesian coordinates. **[3 marks]**

   Show that if $\mathbf{C}$ is a constant vector,

$$\nabla \times (\mathbf{C} \times \mathbf{A}) = \mathbf{C}(\nabla \cdot \mathbf{A}) - (\mathbf{C} \cdot \nabla)\mathbf{A}.$$ **[5 marks]**

   Hence or otherwise show that if $\mathbf{B}$ is a second vector point function,

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}(\nabla \cdot \mathbf{B}) - (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} - \mathbf{B}(\nabla \cdot \mathbf{A}).$$ **[4 marks]**

   Verify this identity by direct calculation if

$$\mathbf{A} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$
$$\mathbf{B} = x\hat{\imath} - y\hat{\jmath} - z\hat{k}.$$ **[8 marks]**

2. Give a coordinate-independent definition of the curl of a vector field **A**. **[3 marks]**

¿From your definition prove Stokes' Theorem

$$\int_S curl\mathbf{A}\cdot\hat{n}\; dS = \int_\gamma \mathbf{A}\cdot d\mathbf{r},$$

where the closed contour γ is along the boundary of the surface S, $d\mathbf{r}$ is a line element along γ, and $\hat{n}$ is a unit vector normal to S whose direction is determined by the motion of a right-handed screw rotated in the direction of γ. **[6 marks]**

Verify Stokes' Theorem by direct calculation when

$$\mathbf{A} = z^3\hat{\imath} + 5x\hat{\jmath} + 3xz^2\hat{k}$$

and S is the octant of a sphere bounded by the planes $x = 0$, $y = 0$, and the surface $x^2 + y^2 + z^2 = a^2$, $(x \geq 0, y \geq 0, z \geq 0,)$ and γ its boundary in the xy-plane. **[11 marks]**

3. A string of length l is fixed at its ends, $x = 0$ and $x = l$, and the displacement $y(x, t)$ obeys the wave equation

$$\frac{\partial^2 y}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0,$$

where c is a real constant. The string is released from rest at time $t = 0$ by giving it a small displacement at the centre in the y-direction.

By applying the method of separation of the variables show that

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{l})\cos(\frac{n\pi ct}{l}),$$

where the A_n are constants. **[12 marks]**

If the displacement at $t = 0$ is of the form

$$y(x,0) = \begin{cases} \alpha x, & \text{if } 0 \le x \le \frac{l}{2}; \\ \alpha(l - x), & \text{if } \frac{l}{2} \le x \le l. \end{cases}$$

where α is a constant, show that the subsequent displacement is given by

$$y(x,t) = \frac{4\alpha l}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}.$$ **[8 marks]**

$$[\text{ Assume that, for integer n, m, } \int_0^l \sin(n\pi x/l)\sin(m\pi x/l)dx = \frac{l}{2}\delta_{nm}.\ ]$$

4. Determine the nature of the singularity at the point $x = 0$ in the second order linear ordinary differential equation

$$x\frac{d^2y}{dx^2} + (q + x)\frac{dy}{dx} - \alpha y = 0.$$

where q and α are positive constants. **[3 marks]**

If the equation has a series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+c} \qquad (a_0 \neq 0)$$

show that $c = 0$ or $1 - q$.

Show further that when $c = 0$,

$$(n + q)(n + 1)a_{n+1} = (\alpha - n)a_n$$ **[7 marks]**

Show that if $\alpha = m$, a positive integer, a polynomial solution is obtained. **[2 marks]**

Write down this solution if $m = 3, q = 3$ and $a_0 = 1$. **[4 marks]**

Show that for general integer m,

$$y(x) = a_0 \sum_{n=0}^{m} \frac{m!(q-1)!x^n}{(m-n)!n!(q+n-1)!}$$ **[4 marks]**

5. (a) A function $f(x)$, periodic in $-\pi < x < \pi$ with interval 2π, has a Fourier series expansion of the form

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}[a_n \cos nx + b_n \sin nx]$$

Show that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \; dx, \qquad n \geq 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \; dx, \qquad n \geq 1$$

[4 marks]

Explain what is meant by the statements that a function $f(x)$ is

(i) an even function of x,

(ii) an odd function of x. **[2 marks]**

Show that if $g(x)$, $(-\pi < x < \pi)$ is odd, then

$$\int_{-\pi}^{\pi} g(x) dx = 0.$$

[2 marks]

Show that an even function of x has a Fourier cosine series only. **[2 marks]**

(b) The generating function for the Legendre polynomials $P_l(x)$, $-1 \leq x \leq 1$, is

$$(1 - 2xt + t^2)^{-1/2} = \sum_{l=0}^{\infty} P_l(x) t^l \qquad (-1 < t < 1.)$$

By expanding the left hand side in a Maclaurin series in t, show that

$$P_0(x) = 1; \quad P_1(x) = x; \quad P_2(x) = \frac{1}{2}(3x^2 - 1).$$

[4 marks]

A function $f(x)$, defined in $-1 \leq x \leq 1$, may be expanded in the form

$$f(x) = \sum_{l=0}^{\infty} a_l P_l(x).$$

Given that

$$\int_{-1}^{1} P_l(x) P_k(x) dx = \frac{2\delta_{lk}}{(2l + 1)},$$

Obtain an integral expression for the coefficients a_l. **[3 marks]**

Find a_0, a_1 and a_2 if $f(x) = \sin \pi x$. **[3 marks]**

6. Define the **cofactors of order (n-1)**, $\alpha_{\mathbf{ij}}^{(\mathbf{n-1})}$, and the **Adjoint Matrix**, $\mathbf{A^{adj}}$, of a square matrix $\mathbf{A}$ of order n.

Express the inverse of $\mathbf{A}$, $\mathbf{A}^{-1}$, in terms of $\mathbf{A}^{adj}$. **[4 marks]**

Find the inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} \lambda & 1 & -\mu \\ -2 & 3 & 2 \\ -1 & 1 & 1 \end{pmatrix}$$

where $\lambda \neq \mu$. **[7 marks]**

Verify your answer by evaluating $\mathbf{AA}^{-1}$. **[2 marks]**

<u>Hence</u> find a column vector $\mathbf{x}$ such that

$$\mathbf{Ax} = \begin{pmatrix} \beta \\ -\beta \\ \beta \end{pmatrix}.$$

where β is a constant. **[3 marks]**

Discuss the solutions when

(a) $\lambda = \mu, \beta \neq 0$.

(b) $\lambda = \mu, \beta = 0$. **[2 marks]**

In case (b), show that it is possible to find a real solution vector $\mathbf{x}$, independent of λ, satisfying $\mathbf{x}^T\mathbf{x} = 1$, and find the vector $\mathbf{x}$. **[2 marks]**

7. The equations of motion of a coupled mechanical system are given by

$$\frac{dx_1}{dt} = -4x_1 + 3x_2$$

$$\frac{dx_2}{dt} = 3x_1 - 12x_2$$

(a) Express the equations in matrix form

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}$$

where $\mathbf{x}$ is a column vector with elements $x_i, i = 1, 2$ and $\mathbf{A}$ is a (2×2) matrix.

(b) Find the eigenvalues and corresponding normalised eigenvectors of $\mathbf{A}$. **[9 marks]**

(c) By making the substitution $\mathbf{x} = \mathbf{U}\mathbf{y}$, where $\mathbf{U}$ is a (2×2) matrix independent of t, show that

$$\frac{d\mathbf{y}}{dt} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}\mathbf{y}.$$ **[2 marks]**

Determine a Unitary matrix $\mathbf{U}$ and a diagonal matrix $\mathbf{h}$, such that $\mathbf{h} = \mathbf{U}^{-1}\mathbf{A}\mathbf{U}$ is a diagonal matrix. **[3 marks]**

(d) Obtain the general solution for $\mathbf{y}$. **[3 marks]**

Hence, given that at time $t = 0$, $x_1(0) = x_0$ and $x_2(0) = 0$ show that at future times

$$x_1(t) = \frac{x_0}{10}(9e^{-3t} + e^{-13t})$$
$$x_2(t) = \frac{3x_0}{10}(e^{-3t} - e^{-13t})$$

[3 marks]

END OF PAPER