UNIVERSITY OF LONDON
(University College London)

PHYSICS 2B21: Mathematical Methods in Physics and Astronomy

xx-MAY-03

Answer FIVE questions only.

Numbers in square brackets show the provisional allocation of marks per sub-section of the question.

1. State Stokes' theorem in integral form. **[2 marks]**

   In plane polar coordinates, where the Cartesian components are given by $x = r\cos\theta$ and $y = r\sin\theta$, show that the unit vector in the θ direction is

   $$\underline{\hat{e}}_\theta = -\sin\theta\,\underline{\hat{e}}_x + \cos\theta\,\underline{\hat{e}}_y\ .$$ **[2 marks]**

   Calculate the line integral $I = \oint_\gamma \underline{W}\cdot\underline{ds}$ of the vector

   $$\underline{W} = (x+y)\,\underline{\hat{e}}_x + xy^2\,\underline{\hat{e}}_y + x^2\,\underline{\hat{e}}_z$$

   anticlockwise around the figure shown in the plane $z = 0$. This consists (a) of the axis $y = 0$, (b) a quarter-circle of radius 1, with its centre at the origin, and (c) the axis $x = 0$. **[10 marks]**

   Evaluate $curl\,\underline{W} = \underline{\nabla}\times\underline{W}$ and hence verify Stokes' theorem by integrating $curl\,\underline{W}$ over the area of the quarter-circle in the x-y plane. **[6 marks]**

   Note that the surface element in plane polar coordinates is

   $$dS = r\,dr\,d\theta\ .$$

2. (a) In spherical polar coordinates ($x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$), the line element is given by

$$d\underline{r} = dr\,\underline{\hat{e}}_r + r\,d\theta\,\underline{\hat{e}}_\theta + r\sin\theta\,d\phi\,\underline{\hat{e}}_\phi\,,$$

where $\underline{\hat{e}}_r$, $\underline{\hat{e}}_\theta$, and $\underline{\hat{e}}_\phi$ are basis vectors in the directions of increasing r, θ and ϕ respectively. Show that in these coordinates

$$\underline{\nabla}\,f = \left(\frac{\partial f}{\partial r}\right)\underline{\hat{e}}_r + \frac{1}{r}\left(\frac{\partial f}{\partial \theta}\right)\underline{\hat{e}}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial f}{\partial \phi}\right)\underline{\hat{e}}_\phi\,.$$ **[4 marks]**

If $f = x^2 + y^2 - 2z^2$, evaluate $\underline{\nabla} f$ in both Cartesian and spherical polar coordinates and show that they are equal. **[6 marks]**

Note that the relation between the basis vectors in spherical polar and Cartesian coordinates is:

$$\begin{aligned}
\underline{\hat{e}}_r &= \sin\theta\cos\phi\,\underline{\hat{e}}_x + \sin\theta\sin\phi\,\underline{\hat{e}}_y + \cos\theta\,\underline{\hat{e}}_z\,,\\
\underline{\hat{e}}_\theta &= \cos\theta\cos\phi\,\underline{\hat{e}}_x + \cos\theta\sin\phi\,\underline{\hat{e}}_y - \sin\theta\,\underline{\hat{e}}_z\,,\\
\underline{\hat{e}}_\phi &= -\sin\phi\,\underline{\hat{e}}_x + \cos\phi\,\underline{\hat{e}}_y\,.
\end{aligned}$$

(b) The potential $V(r,\theta)$ in plane polar coordinates satisfies the equation

$$\underline{\nabla}^2 V(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial}{\partial r}V(r,\theta)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}V(r,\theta) = 0\,.$$

By searching for a solution in the separable form, $V(r,\theta) = R(r)\times\Theta(\theta)$ show that the general solution in the region $0 \le \theta \le 2\pi$ is

$$V(r,\theta) = A + B\,\ell n\,r + \sum_{n=1}^{\infty}\left(C_n r^n + \frac{D_n}{r^n}\right)\left(E_n\cos n\theta + F_n\sin n\theta\right)\,.$$ **[7 marks]**

If the potential on the ring $r = a$ is given by $V(a,\theta) = V_0\cos\theta$, evaluate the potential in the regions $0 \le r \le a$ and $a \le r < \infty$. **[3 marks]**

3. (a) The matrices $\underline{A}$, $\underline{B}$, and $\underline{D}$ are related by $\underline{D} = \underline{A}\,\underline{B}$. Given that

$$\underline{A} = \begin{pmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \\ 3 & 3 & 2 \end{pmatrix} \qquad \text{and} \qquad \underline{D} = \begin{pmatrix} 10 & -6 & 6 \\ 9 & -6 & 5 \\ 15 & -10 & 11 \end{pmatrix} ,$$

evaluate $\underline{A}^{-1}$. **[7 marks]**

Hence derive the value of $\underline{B}$. **[3 marks]**

(b) Find the eigenvalues of the matrix

$$\underline{A} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} .$$ **[2 marks]**

Show that $\underline{A}^2 = \underline{I} + 2\underline{A}$ and hence evaluate $\underline{A}^4$. **[3 marks]**

If t_n is defined in terms of the trace of a matrix through

$$t_n = [tr(\underline{A}^n)]^{1/n} ,$$

show that $t_2 \approx 2.4495$ and $t_4 \approx 2.4147$. **[2 marks]**

Why does $t_n \to \sqrt{2}+1$ as $n \to \infty$? **[3 marks]**

4. By writing a square matrix $\underline{A}$ in terms of its matrix of eigenvalues $\underline{\Lambda}$ through $\underline{A} = \underline{R}^{-1}\,\underline{\Lambda}\,\underline{R}$, show that the trace of $\underline{A}$ is equal to the sum of the eigenvalues:

$$tr\{\underline{A}\} = \sum_i A_{ii} = \sum_i \lambda_i .$$ **[3 marks]**

Demonstrate that the eigenvalues λ of the Hermitian matrix

$$\underline{A} = \begin{pmatrix} 1 & i & 3i \\ -i & 1 & -3 \\ -3i & -3 & -3 \end{pmatrix}$$

satisfy the characteristic equation

$$\lambda^3 + \lambda^2 - 24\lambda + 36 = 0 .$$ **[3 marks]**

Prove that one eigenvalue is $\lambda_1 = 2$ and find the other two solutions. **[2 marks]**

Find the three (complex) eigenvectors $\underline{x}_i$, normalised to have unit length, $\underline{x}_i^\dagger\,\underline{x}_i = 1$, where the $\dagger$ denotes Hermitian conjugation. **[9 marks]**

Prove that the eigenvectors are orthogonal,

$$\underline{x}_i^\dagger\,\underline{x}_j = 0 \quad \text{for} \quad i \neq j .$$ **[3 marks]**

5. Show that the second order differential equation

$$(2x + x^2)\,\frac{d^2y}{dx^2} + (1+x)\,\frac{dy}{dx} - p^2 y = 0$$

has two solutions of the form

$$y = \sum_{n=0}^{\infty} a_n\, x^{n+k}\,, \qquad a_0 \neq 0$$

with $k = 0$ or $k = \frac{1}{2}$. **[6 marks]**

Derive the recurrence relation

$$\frac{a_{n+1}}{a_n} = -\frac{(n+k)^2 - p^2}{(n+k+1)(2n+2k+1)}\,.$$ **[4 marks]**

Use the d'Alembert ratio test to determine the range of values of x for which the series converges. **[3 marks]**

In the special case where p is a positive integer, show that the $k = 0$ series terminates at $n = p$. **[3 marks]**

Denote the resulting polynomial by $T_p(x)$. If $T_p(0) = 1$, show that to order x the polynomials satisfy

$$2\,T_p(x)\,T_q(x) = T_{p+q}(x) + T_{p-q}(x)\,,$$

where q is another positive integer with $p \geq q$. **[4 marks]**

6. The function $f(x)$ is periodic with period 2π. In the interval $-\pi < x < +\pi$, it is given by

$$f(x) = \sinh x \, .$$

Is $f(x)$ even or odd? **[1 mark]**

If $f(x)$ has a Fourier series expansion of the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \, ,$$

show, by quoting the orthogonality of the sine and cosine functions, that the Fourier coefficients are given by

$$\begin{aligned} a_n &= 0 \, , \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx \, . \end{aligned}$$

[5 marks]

By using integration by parts twice, or otherwise, show that

$$\int \sin nx \sinh x \, dx = \frac{1}{1+n^2} \left[\cosh x \sin nx - n \cos nx \sinh x\right] + C \, .$$

[4 marks]

For the particular case of $f(x) = \sinh x$, obtain the coefficients b_n and show that its Fourier series is

$$f(x) = \frac{2}{\pi} \sinh \pi \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1} \sin nx \, .$$

[4 marks]

State Parseval's theorem and use it to evaluate

$$\sum_{n=1}^{\infty} \frac{n^2}{(n^2+1)^2} \, .$$

[6 marks]

You may find the following identity useful:

$$\sinh^2 x = \tfrac{1}{2}(\cosh 2x - 1) \, .$$

7. Starting from the differential equation for the Legendre polynomial $P_n(x)$,

$$\frac{d}{dx}\left[(1-x^2)\,\frac{d}{dx}P_n(x)\right] + n(n+1)\,P_n(x) = 0\,,$$

show that the definite integral

$$\int_{-1}^{+1} P_n(x)\,P_m(x)\,dx = 0\,,$$

if n and m are non-negative integers with $n \neq m$. **[10 marks]**

Given that $P_0(x) = 1$, $P_1(x) = x$, and $P_2(x) = \frac{1}{2}(3x^2 - 1)$, show by explicit integration that the above orthogonality relation is satisfied for $n, m \leq 2$. **[3 marks]**

Assuming that for $n = 3$

$$P_3(x) = a\,[x^3 + bx^2 + cx + d]\,,$$

use the orthogonality relation to find the coefficients b, c and d. **[4 marks]**

How can the coefficient a be determined? **[1 mark]**

Show that $a = \frac{5}{2}$. **[2 marks]**

END OF PAPER