UNIVERSITY OF LONDON
(University College London)

PHYSICS 2B21: Mathematical Methods in Physics and Astronomy

20-MAY-02

All questions may be attempted. *Credit will be given for all work done correctly. Numbers in square brackets show the provisional allocation of marks per sub-section of the question.*

1. State the divergence theorem. **[2 marks]**

   Calculate the integral of the divergence of the vector field

   $$\underline{F} = x^2\,\underline{\hat{e}}_x + 4\,\underline{\hat{e}}_y + (z+1)^2\,\underline{\hat{e}}_z$$

   over the volume bounded by $0 \leq z \leq 1$, $x \geq 0$, $y \geq 0$ and the curved surface $x^2 + y^2 \leq 1$. **[8 marks]**

   Calculate explicitly the flux of $\underline{F}$ through the top and bottom, and the flat and curved sides of the quarter-cylinder. Hence verify the divergence theorem in this case. **[10 marks]**

   Note that in cylindrical polar coordinates,

   $$x = r\cos\theta\,, \quad y = r\sin\theta\,, \quad z = z\,,$$

   the volume element is

   $$dV = r\,dr\,d\theta\,dz\,,$$

   and the surface element at constant r is

   $$dS = r\,d\theta\,dz\,.$$

2. (a) By writing the equation in Cartesian coordinates, show that

$$\underline{\nabla} \cdot (\underline{A} \times \underline{B}) = \underline{B} \cdot (\underline{\nabla} \times \underline{A}) - \underline{A} \cdot (\underline{\nabla} \times \underline{B}) \, ,$$

where $\underline{A}$ and $\underline{B}$ are vector functions of (x, y, z). **[6 marks]**

Evaluate the right hand side of the identity for the vectors $\underline{A} = (x, y, z)$ and $\underline{B} = (y, z, x)$. **[4 marks]**

(b) A function $u(x, y)$ of two independent variables x and y satisfies the first order partial differential equation

$$x \frac{\partial u}{\partial x} - \tfrac{1}{2} y \frac{\partial u}{\partial y} = 0 \, .$$

By first looking for a separable solution of the form $u(x, y) = X(x) \times Y(y)$, find the general solution of the equation. **[7 marks]**
Determine the $u(x, y)$ which satisfies the boundary condition $u(1, y) = 1 + \sin y$. **[3 marks]**

3. (a) Write the simultaneous equations

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 2 \, , \\ 2x_1 - x_2 + x_3 &= 3 \, , \\ 3x_1 + 5x_2 + 2x_3 &= 1 \end{aligned}$$

in matrix form $\underline{A}\,\underline{x} = \underline{b}$, where $\underline{x}$ and $\underline{b}$ are column vectors. **[2 marks]**

Find the inverse matrix $\underline{A}^{-1}$. **[6 marks]**

Use $\underline{A}^{-1}$ to solve for x_1, x_2, and x_3. **[2 marks]**

(b)
- If † denotes Hermitian conjugation, show that

$$(\underline{A}\,\underline{B})^\dagger = \underline{B}^\dagger \underline{A}^\dagger \, .$$
[3 marks]

- The trace of a matrix $\underline{C}$ is the sum of its diagonal elements,

$$Tr\{\underline{C}\} = \sum_i C_{ii} .$$

By writing out the matrix multiplication explicitly in terms of components, show that for any matrix $\underline{S}$ the trace of $\underline{C} = \underline{S}^\dagger \underline{S}$ can never be negative. **[3 marks]**
Verify this result explicitly in the case where

$$\underline{S} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$
[4 marks]

4. A real quadratic form F is defined by

$$F = \underline{X}^T \, \underline{A} \, \underline{X} = (x_1, x_2, x_3) \begin{pmatrix} 1 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} .$$

Show that two of the eigenvalues of the matrix $\underline{A}$ are $\lambda_1 = 1$ and $\lambda_2 = 3$ and determine the third one. **[4 marks]**

Derive the three corresponding normalised eigenvectors and, by working out their scalar products, show that they are mutually perpendicular. **[8 marks]**

By performing an orthogonal transformation to a new vector $\underline{y}$,

$$\underline{x} = \underline{R} \, \underline{y} \, ,$$

with

$$\underline{R}^T \, \underline{R} = \underline{I} \, ,$$

find the form of $\underline{R}$ such that F can be written in the diagonal form **[2 marks]**

$$F = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 \, . \qquad (*)$$

Express y_1, y_2 and y_3 in terms of x_1, x_2 and x_3. **[4 marks]**

By substituting these expressions for the y_i into equation (*), show that one recovers the original form for F in terms of the x_i. **[2 marks]**

5. Prove that the second order differential equation

$$x^2 \frac{d^2 y}{dx^2} + 2x^2 \frac{dy}{dx} - 2y = 0 \, ,$$

has two solutions of the form

$$y = \sum_{n=0}^{\infty} a_n \, x^{n+k} \, , \qquad a_0 \neq 0$$

with $k = -1$ or $k = 2$. **[6 marks]**

Show that for both series the ratio

$$\frac{a_{n+1}}{a_n} = -\frac{2(n+k)}{(n+k+2)(n+k-1)} \, .$$ **[4 marks]**

Show that the series expansion for the $k = -1$ solution terminates at $n = 1$ and verify explicitly that the resultant expression does satisfy the differential equation. **[4 marks]**

Show that the $k = 2$ series converges for all values of x, **[3 marks]**
and that the first two terms are proportional to the series expansion of

$$y = \left(1 + \frac{1}{x}\right) e^{-2x} + \left(1 - \frac{1}{x}\right) \, .$$ **[3 marks]**

6. The function $f(x)$ is periodic with period 2π. In the interval $-\pi < x < +\pi$, it is given by

$$f(x) = \sin(\tfrac{1}{2}x) \; .$$

Is $f(x)$ even or odd? **[1 mark]**

If $f(x)$ has a Fourier series expansion of the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \; ,$$

show, by quoting the orthogonality of the sine and cosine functions, that the Fourier coefficients are given by

$$\begin{aligned} a_n &= 0 \, , \\ b_n &= \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx \; . \end{aligned}$$

[5 marks]

For the particular $f(x)$, obtain the coefficients b_n and show that the Fourier series is

$$f(x) = \frac{8}{\pi}\sum_{n=1}^{\infty} (-1)^n \frac{n}{(1-4n^2)} \sin nx \; .$$

[6 marks]

Evaluate the Fourier series at $x = \pi$ and comment on the answer. **[2 marks]**

State Parseval's theorem and use it to evaluate

$$\sum_{n=1}^{\infty} \frac{n^2}{(4n^2-1)^2} \; .$$

[6 marks]

N.B. You may assume the relations

$$\sin A \sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]$$

and

$$\int_{-\pi}^{+\pi} \sin nx \, \sin mx \, dx = \int_{-\pi}^{+\pi} \cos nx \, \cos mx \, dx = \pi \, \delta_{nm} \; ,$$

where n and m are positive integers.

7. The definite integral of two Legendre polynomials

$$\int_{-1}^{+1} P_n(x)\, P_m(x)\, dx$$

vanishes unless $n = m$. By using the generating function for $t \le 1$,

$$g(x,t) \equiv (1 - 2x\,t + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x)\, t^n \ ,$$

show that, when $n = m$,

$$\int_{-1}^{+1} [P_n(x)]^2 \ dx = \frac{2}{2n+1} \ .$$ **[9 marks]**

The polynomials satisfy the recurrence relations

$$\begin{aligned} (2n+1)x\,P_n(x) &= (n+1)P_{n+1}(x) + nP_{n-1} \ , \\ (2n+1)P_n(x) &= P'_{n+1}(x) - P'_{n-1}(x) \ . \end{aligned}$$

Use these relations, together with normalisation integrals, to evaluate

$$\int_{-1}^{+1} P_{n+1}(x)\, x\, P_n(x)\, dx$$ **[3 marks]**

and show that

$$\int_{-1}^{+1} P'_{n+1}(x)\, P_n(x)\, dx = 2 \ .$$ **[4 marks]**

Verify both relations by explicit integration for the case of $n = 1$. **[4 marks]**

You may assume that

$$P_0(x) = 1\,, \qquad P_1(x) = x\,, \qquad P_2(x) = \tfrac{1}{2}(3x^2 - 1)\ .$$

END OF PAPER