UNIVERSITY OF LONDON
(University College London)

PHYSICS 2B21: Mathematical Methods in Physics and Astronomy

26-MAY-00

All questions may be attempted. *Credit will be given for all work done correctly. Numbers in square brackets show the provisional allocation of marks per sub-section of the question.*

1. State the divergence theorem. **[2 marks]**

Calculate the integral of the divergence of the vector field

$$\underline{F} = xy\,\hat{\underline{e}}_x + yz^2\,\hat{\underline{e}}_y + \hat{\underline{e}}_z$$

over the volume of the hemisphere defined by $x^2 + y^2 + z^2 \leq 16$ and $z \geq 0$. **[8 marks]**

Write down the z and the radial components of $\underline{F}$ and use them to calculate explicitly the flux of $\underline{F}$ through the base and the curved surface of the hemisphere. Hence verify the divergence theorem in this case. **[10 marks]**

Note that in spherical polar coordinates

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

and the radius vector points in the direction

$$\hat{r} = \sin\theta\cos\phi\,\hat{\underline{e}}_x + \sin\theta\sin\phi\,\hat{\underline{e}}_y + \cos\theta\,\hat{\underline{e}}_z\ .$$

The element of area perpendicular to the radius vector is

$$dA_r = r^2\ \sin\theta\,d\theta\,d\phi\ ,$$

and the corresponding element of volume is

$$dV = r^2\,dr\ \sin\theta\,d\theta\,d\phi\ .$$

Integrals of the type $\int \sin^{2n+1}\theta\ \cos^{2m}\theta\,d\theta$ (m and n integers) can be evaluated by using the substitution $t = \cos\theta$, $dt = -\sin\theta\,d\theta$.

2. (a) By evaluating both sides explicitly in Cartesian coordinates, verify the identity

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) - \nabla^2 \underline{A}$$

for the vector field

$$\underline{A} = x^2 yz\, \hat{\underline{e}}_x + xyz^2\, \hat{\underline{e}}_y + y^2 z\, \hat{\underline{e}}_z \,.$$ **[10 marks]**

(b) A drumhead consists of a circular membrane attached to a rigid support along the circumference $r = a$. The vibrations are governed by the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial Z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2} = \frac{1}{v^2}\frac{\partial^2 Z}{\partial t^2}\,,$$

where Z is the displacement from equilibrium at polar coordinate $(r,\, \theta)$ and time t, and v is a constant. By assuming a solution of the form

$$Z(r, \theta, t) = R(r) \times \Theta(\theta) \times T(t)\,,$$

derive ordinary differential equations for $R(r)$, $\Theta(\theta)$, and $T(t)$. **[4 marks]**

Show that solutions which have $Z = 0$ at $t = 0$ are of the form

$$Z = R_n(kr)\sin(kvt)\left[a_n \cos n\theta + b_n \sin n\theta\right],$$

where n is an integer. **[5 marks]**

How can one find information on the possible values of k? **[1 mark]**

3. (a) Write the simultaneous equations

$$\begin{aligned} 3x_1 - 2x_2 - x_3 &= 4\,, \\ 2x_1 + x_2 + 2x_3 &= 10\,, \\ x_1 + 3x_2 - 4x_3 &= 5 \end{aligned}$$

in matrix form $\underline{A}\,\underline{x} = \underline{b}$, where $\underline{x}$ and $\underline{b}$ are column vectors. **[2 marks]**

Find the inverse matrix $\underline{A}^{-1}$. **[6 marks]**

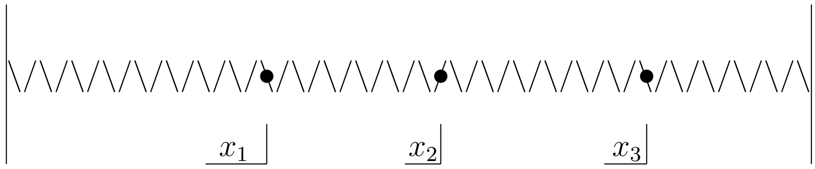
Use $\underline{A}^{-1}$ to solve for x_1, x_2, and x_3. **[2 marks]**

(b) For any real parameter α, the matrix

$$\underline{H} = \alpha \begin{pmatrix} \cos\theta & i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix}$$

is Hermitian. By expanding the exponential in a power series, verify to order α^2 that $\underline{U} = \exp(i\underline{H})$ is unitary ($\underline{U}^\dagger\, \underline{U} = \underline{I}$). **[10 marks]**

4. Three particles of equal masses, attached to a light spring, can move in a straight line, as illustrated in the diagram.



The equations of motion may be written in matrix form

$$\frac{d^2\underline{x}}{dt^2} = \underline{A}\,\underline{x}\,,$$

where

$$\underline{A} = \begin{pmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{pmatrix}$$

and $\underline{x}$ is a column vector of the displacements x_i.

Show that the eigenvalues of $\underline{A}$ are $\lambda_1 = -1$, $\lambda_2 = -3$, and $\lambda_3 = -4$. **[5 marks]**

Find the corresponding normalised eigenvectors. **[7 marks]**

By making the transformation $\underline{x} = \underline{R}\,\underline{y}$, where $\underline{R}$ is an orthogonal matrix independent of t, the equations of motion may be transformed to

$$\frac{d^2\underline{y}}{dt^2} = \Lambda\,\underline{y}\,,$$

where Λ is the diagonal matrix of the eigenvalues of $\underline{A}$. Find the general solutions for the y_i as functions of time. **[2 marks]**

Determine the elements of the matrix $\underline{R}$ and hence solve for y_2 in terms of the x_i and give a physical interpretation of this normal mode. **[6 marks]**

5. Show that the second order differential equation

$$x\,\frac{d^2y}{dx^2} + (1 + p - x)\,\frac{dy}{dx} + by = 0\,,$$

where b and p are constants with $p > 0$, has two solutions of the form

$$y = \sum_{n=0}^{\infty} a_n\, x^{n+k}\,, \qquad a_0 \neq 0$$

with $k = 0$ or $k = -p$. **[6 marks]**

Derive the recurrence relation

$$\frac{a_{n+1}}{a_n} = \frac{n + k - b}{(n + k + 1)(n + k + 1 + p)}\,.$$ **[4 marks]**

Use the d'Alembert ratio test to show that both series converge for all values of x. **[4 marks]**

In the special case of $b = m$, a positive integer, show that the series with $k = 0$ terminates at $n = m$ to yield a polynomial solution. **[2 marks]**

By differentiating the original differential equation with respect to x, show that if y is a solution for particular values of m and p, then dy/dx is also a solution to the equation with $m \to m - 1$ and $p \to p + 1$. **[4 marks]**

6. The function $f(x)$ is periodic with period 2π. In the interval $-\pi < x < +\pi$, it is given by

$$f(x) = \cos(\tfrac{1}{2}x) \, .$$

Is $f(x)$ even or odd? **[2 marks]**

If $f(x)$ has a Fourier series expansion of the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \, ,$$

show, by using the orthogonality of the sine and cosine functions, that the Fourier coefficients are given by

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\, \cos nx \, dx \, , \\ b_n &= 0 \, . \end{aligned}$$

[6 marks]

Evaluate the coefficients a_n and show that the Fourier series for $f(x)$ is

$$f(x) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{(1-4n^2)} \cos nx \, .$$

[6 marks]

State Parseval's theorem and use it to evaluate

$$\sum_{n=0}^{\infty} \frac{1}{(4n^2-1)^2} \, .$$

[6 marks]

N.B. You may assume the relation

$$\cos A \cos B = \tfrac{1}{2}[\cos(A+B) + \cos(A-B)] \, .$$

7. The Legendre polynomials $P_n(x)$ satisfy the differential equation

$$\frac{d}{dx}\left[(1-x^2)\,\frac{d}{dx}P_n(x)\right] + n(n+1)\,P_n(x) = 0\,,$$

where n is a non-negative integer. Deduce the orthogonality relation

$$\int_{-1}^{+1} P_m(x)P_n(x)\,dx = 0\,, \qquad (n \neq m)\,.$$ **[10 marks]**

Given that

$$\int_{-1}^{+1} [P_n(x)]^2\;dx = \frac{2}{2n+1}\,,$$

and that a function $f(x)$ can be expressed as a series of Legendre polynomials,

$$f(x) = \sum_{n=0}^{\infty} a_n\,P_n(x)\,, \qquad (-1 \leq x \leq +1)\,,$$

obtain a formula for the coefficients a_n. **[4 marks]**

If the function $f(x)$ is defined by

$$f(x) = \frac{1}{(x+c)^{1/2}}\,, \qquad (c > 1)\,,$$

determine the coefficients a_0 and a_1. **[6 marks]**

 END OF PAPER