UNIVERSITY OF LONDON
(University College London)

PHYSICS 2B21: Mathematical Methods in Physics and Astronomy

Mid–Sessional Examination

Friday 12 December 2003: 10.30 to 12.30

Answer FOUR questions only.

1. (a) If ϕ is a scalar function, $\underline{S}$ a vector function, and $\underline{C}$ a constant vector, show by writing out both sides explicitly in Cartesian coordinates that

$$\underline{\nabla} \cdot (\phi\, \underline{S}) = \phi\, (\underline{\nabla} \cdot \underline{S}) + (\underline{\nabla}\phi) \cdot \underline{S}\,,$$ **[3 marks]**

$$\underline{\nabla} \times (\phi\, \underline{S}) = \underline{\nabla}\phi \times \underline{S} + \phi\, (\underline{\nabla} \times \underline{S})\,,$$ **[3 marks]**

and also that

$$\underline{C} \times (\underline{\nabla} \times \underline{S}) = \underline{\nabla}\, (\underline{C} \cdot \underline{S}) - (\underline{C} \cdot \underline{\nabla})\, \underline{S}\,.$$ **[4 marks]**

(b) In spherical polar coordinates ($x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$), the line element is given by

$$d\underline{r} = dr\, \hat{\underline{e}}_r + r\, d\theta\, \hat{\underline{e}}_\theta + r\sin\theta\, d\phi\, \hat{\underline{e}}_\phi\,,$$

where the basis vectors are

$$\begin{aligned} \hat{\underline{e}}_r &= \sin\theta\cos\phi\, \hat{\underline{e}}_x + \sin\theta\sin\phi\, \hat{\underline{e}}_y + \cos\theta\, \hat{\underline{e}}_z\,, \\ \hat{\underline{e}}_\theta &= \cos\theta\cos\phi\, \hat{\underline{e}}_x + \cos\theta\sin\phi\, \hat{\underline{e}}_y - \sin\theta\, \hat{\underline{e}}_z\,, \\ \hat{\underline{e}}_\phi &= -\sin\phi\, \hat{\underline{e}}_x + \cos\phi\, \hat{\underline{e}}_y\,. \end{aligned}$$

Show that in these coordinates

$$\underline{\nabla} f = \left(\frac{\partial f}{\partial r}\right) \hat{\underline{e}}_r + \frac{1}{r}\left(\frac{\partial f}{\partial \theta}\right) \hat{\underline{e}}_\theta + \frac{1}{r\sin\theta}\left(\frac{\partial f}{\partial \phi}\right) \hat{\underline{e}}_\phi\,.$$ **[4 marks]**

If $f = x^2 + y^2$, evaluate $\underline{\nabla} f$ in both Cartesian and spherical polar coordinates and show that they are equal in magnitude and direction. **[6 marks]**

2. (a) The matrices $\underline{A}$, $\underline{B}$, and $\underline{D}$ are related by $\underline{D} = \underline{B}\,\underline{A}$. Given that

$$\underline{A} = \begin{pmatrix} 3 & 1 & -3 \\ 1 & 4 & 2 \\ -3 & 2 & 5 \end{pmatrix} \quad \text{and} \quad \underline{D} = \begin{pmatrix} -2 & 12 & 11 \\ 14 & 17 & -3 \\ -5 & 16 & 19 \end{pmatrix},$$

evaluate $\underline{A}^{-1}$. **[7 marks]**

Hence derive the value of $\underline{B}$. **[3 marks]**

(b) The Pauli matrices

$$\underline{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \underline{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \underline{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are of great importance in the description of spin-$\frac{1}{2}$ particles in quantum mechanics. What special property do these matrices have in common? **[1 mark]**

Show that for $i = 1, 2, 3$

$$\underline{\sigma}_i\,\underline{\sigma}_i = \underline{I}\,,$$ **[2 marks]**

where $\underline{I}$ is the 2×2 unit matrix.

Evaluate $\underline{\sigma}_1\,\underline{\sigma}_2$ and $\underline{\sigma}_2\,\underline{\sigma}_1$ in terms of $\underline{\sigma}_3$. **[2 marks]**

By expanding the left hand side in a power series, prove that

$$\exp\left[i\alpha\,\underline{\sigma}_2\right] = \underline{I}\cos\alpha + i\underline{\sigma}_2\sin\alpha\,,$$ **[5 marks]**

where α is a real angle.

3. The matrix $\underline{A}$ is given by

$$\underline{A} = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -2 & -1 \\ 2 & -1 & 1 \end{pmatrix} .$$

Verify that one of the eigenvalues is $\lambda_1 = 0$ and that the corresponding *normalised* eigenvector is $\underline{v}_1 = \dfrac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. **[5 marks]**

Find the other two eigenvalues λ_2 and λ_3 and the associated normalised eigenvectors $\underline{v}_2$ and $\underline{v}_3$. **[8 marks]**

Show that these eigenvectors are mutually orthogonal and that, up to a possible overall sign, **[3 marks]**

$$\underline{v}_3 = \pm(\underline{v}_1 \times \underline{v}_2) .$$ **[2 marks]**

Explain the origin of these last two results. **[2 marks]**

4. The variable y satisfies the second order differential equation

$$(1-x^2)\,\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} - y = 0\,.$$

If y is expanded as the power series

$$y = \sum_{n=0}^{\infty} a_n\, x^{n+k}\,, \qquad a_0 \neq 0\,,$$

show that there are two solutions of the indicial equation with $k = 0$ and $k = 1$. **[6 marks]**

Find the recurrence relation between a_{n+2} and a_n in both series. **[4 marks]**

Use the d'Alembert ratio test to show that both series converge for $-1 < x < +1$. **[3 marks]**

Show by explicit differentiation that

$$y = \cosh(\arcsin x)$$

is a solution of the original differential equation. **[5 marks]**

Hence explain why the range of convergence deduced using the ratio test is not unexpected. **[2 marks]**

$$\left[\text{Note that}\quad \frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}\,\cdot\right]$$

5. The function $f(x)$, which is periodic with period 2π, has a Fourier series expansion of the form

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \,.$$

Show, by quoting the orthogonality of the sine and cosine functions, that the Fourier coefficients are given by

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\cos nx\,dx\,, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin nx\,dx\,. \end{aligned}$$

[6 marks]

In the interval $-\pi < x < +\pi$, the function is given by

$$f(x) = e^{\lambda x}\,,$$

where λ is a real constant.

Show, by writing $e^{inx} = \cos nx + i\sin nx$, or by integration by parts twice, that

$$\begin{aligned} \int_{-\pi}^{\pi} e^{\lambda x}\cos nx\,dx &= 2(-1)^n \sinh\lambda\pi\,\frac{\lambda}{\lambda^2+n^2} \\ \int_{-\pi}^{\pi} e^{\lambda x}\sin nx\,dx &= -2(-1)^n \sinh\lambda\pi\,\frac{n}{\lambda^2+n^2}\,, \end{aligned}$$

where n is an integer. **[4 marks]**

Hence write down the Fourier series for this function. **[2 marks]**

State Parseval's theorem for a real Fourier series and use it with this function to evaluate

$$\sum_{n=1}^{\infty} \frac{\lambda^2}{n^2+\lambda^2}\,.$$

[8 marks]

6. (a) A drumhead consists of a circular membrane attached to a rigid support along the circumference $r = a$. The vibrations are governed by the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial Z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 Z}{\partial \theta^2} = \frac{1}{v^2}\frac{\partial^2 Z}{\partial t^2}\ ,$$

where Z is the displacement from equilibrium at polar coordinate $(r,\,\theta)$ and time t, and v is a constant. By assuming a solution of the form

$$Z(r,\theta,t) = R(r) \times \Theta(\theta) \times T(t)\ ,$$

derive ordinary differential equations for $R(r)$, $\Theta(\theta)$, and $T(t)$. **[4 marks]**

Show that solutions which have $Z = 0$ at $t = 0$ are of the form

$$Z = R_n(kr)\sin(kvt)\left[a_n\cos n\theta + b_n\sin n\theta\right]\ ,$$

where n is an integer. **[5 marks]**

How can one find information on the possible values of k? **[1 mark]**

(b) The definite integral of two Legendre polynomials

$$\int_{-1}^{+1} P_n(x)\,P_m(x)\,dx = \frac{2}{(2n+1)}\,\delta_{nm}\ .$$

Use the recurrence relations

$$\begin{aligned}(2n+1)\,x\,P_n(x) &= (n+1)P_{n+1}(x) + nP_{n-1}(x)\ ,\\ (2n+1)P_n(x) &= P'_{n+1}(x) - P'_{n-1}(x)\ ,\end{aligned}$$

together with the above definite integral, to show that

$$\int_{-1}^{+1} P_{n+1}(x)\,x\,P_n(x)\,dx = \frac{2(n+1)}{(2n+1)(2n+3)}$$ **[3 marks]**

and

$$\int_{-1}^{+1} P'_{n+1}(x)\,P_n(x)\,dx = 2\ .$$ **[3 marks]**

Verify both relations by explicit integration for the case of $n = 1$. **[4 marks]**

You may assume that

$$P_0(x) = 1\ ,\qquad P_1(x) = x\ ,\qquad P_2(x) = \tfrac{1}{2}(3x^2 - 1)\ .$$

MERRY CHRISTMAS !!!