1B45 Mathematical Methods Problem Class 2 2005/2006

Week starting Monday 31st. October

Solutions

1.

(a)

$$\text{We have } \; y = x^2 + 2 + x^{-2} \;\; \text{and} \;\; \frac{dy}{dx} = 2x - \frac{2}{x^3} \; .$$

(b)

$$\text{We have } \; y = x^{\frac{1}{2}} + x^{-\frac{1}{2}} \;\; \text{and} \;\; \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}} = \frac{1}{2\sqrt{x}}\left(1 - \frac{1}{x}\right) \; .$$

(c)

$$\text{We have } \; y = (x^{\frac{1}{2}}+1)(x^{\frac{1}{2}}-1)^{-1} \;\; , \;\; \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(x^{\frac{1}{2}}-1)^{-1} + (x^{\frac{1}{2}}+1)(-1)(x^{\frac{1}{2}}-1)^{-2}\frac{1}{2}x^{-\frac{1}{2}}$$

$$\text{and } \; \frac{dy}{dx} = \frac{1}{2\sqrt{x}}\frac{(\sqrt{x}-1) - (\sqrt{x}+1)}{(\sqrt{x}-1)^2} = \frac{1}{\sqrt{x}}\frac{-1}{(\sqrt{x}-1)^2} \; .$$

(d)

First, if $y = sin^{-1}x$ then

$$x = sin\; y \; , \; dx = cos\; y\; dy = (\sqrt{(1 - sin^2 y})dy = \sqrt{(1 - x^2}\; dy \;\; \text{and} \;\; \frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2)}} \; .$$

$$\text{Then } \; \frac{d}{dx}sin^{-1}x^2 = \frac{1}{\sqrt{(1-x^4)}} \times 2x = \frac{2x}{\sqrt{(1-x^4)}} \; .$$

(e)

$$\text{We have } \; y = ln\,\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}} \;\; \text{and} \;\; \frac{dy}{dx} = \frac{\sqrt{a}-\sqrt{x}}{\sqrt{a}+\sqrt{x}}\,\frac{d}{dx}((\sqrt{a}+\sqrt{x})(\sqrt{a}-\sqrt{x})^{-1})$$

$$= \frac{\sqrt{a}-\sqrt{x}}{\sqrt{a}+\sqrt{x}}\left(\frac{1}{2}x^{-\frac{1}{2}}(\sqrt{a}-\sqrt{x})^{-1} + \frac{(\sqrt{a}+\sqrt{x})(-1)(-\frac{1}{2})x^{-\frac{1}{2}}}{(\sqrt{a}-\sqrt{x})^2}\right)$$

$$= \frac{\sqrt{a}-\sqrt{x}}{\sqrt{a}+\sqrt{x}}\,\frac{1}{2}\,\frac{1}{\sqrt{x}}\frac{\sqrt{a}-\sqrt{x}+\sqrt{a}+\sqrt{x}}{(\sqrt{a}-\sqrt{x})^2} = \frac{1}{\sqrt{x}}\frac{\sqrt{a}}{(a-x)} \; .$$

It is much easier here to start from

$$y = ln\,\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}} = ln\;(\sqrt{a}+\sqrt{x}) - ln\;(\sqrt{a}-\sqrt{x}) \qquad !!!$$

2.

(a)

$$\int x^{-\frac{1}{2}}dx = \frac{1}{\frac{1}{2}}x^{\frac{1}{2}} = 2\sqrt{x}\ .$$

(b)

$$\int (ax+b)^{-\frac{1}{2}}dx = c\quad (ax+b)^{\frac{1}{2}}(\quad \text{c} \quad \text{being a constant to be determined.})$$

Since $\frac{d}{dx}(ax+b)^{\frac{1}{2}} = \frac{a}{2}\,\frac{1}{(ax+b)^{\frac{1}{2}}}$, $c = \frac{2}{a}$ and $\int \frac{dx}{\sqrt{ax+b}} = \frac{2}{a}(ax+b)^{\frac{1}{2}}$.

(c)

We have $x = a\ tan\ y$, $dx = a\ sec^2\ ydy = (1+tan^2 y)dy = \frac{a^2+x^2}{a}dy,$

whence $\frac{d}{dx}tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{(x^2+a^2)}$.

Thus $\int \frac{d}{dx}tan^{-1}\left(\frac{x}{a}\right)\ dx = \int \frac{a\ dx}{(x^2+a^2)}$ and $\int \frac{dx}{(x^2+a^2)} = \frac{1}{a}tan^{-1}\left(\frac{x}{a}\right)$.

Making the substitution we find $\int \frac{dx}{(x^2+a^2)} = \int \frac{a\ sec^2\theta d\theta}{a^2 tan^2\theta + a^2} = \frac{1}{a}\int d\theta = \frac{1}{a}tan^{-1}\left(\frac{x}{a}\right)$.

(d)

We have $\int \frac{dx}{e^x+e^{-x}} = \int \frac{du}{u}\frac{1}{(u+u^{-1})} = \int \frac{du}{(u^2+1)} = tan^{-1}e^x$.

(e)

We have $\frac{2\pi\alpha^2}{s}\int_{-1}^{+1}\frac{dcos\theta}{(1+cos\theta+\frac{2m_e^2}{s})} = \frac{2\pi\alpha^2}{s}\left[ln\ (1+cos\theta+\frac{2m_e^2}{s}\right]_{-1}^{+1} = \frac{2\pi\alpha^2}{s}ln\ \frac{s+m_e^2}{m_e^2}$.

3.

We have $\int_0^\infty e^{-\alpha x}dx = \left[\frac{e^{-\alpha x}}{(-\alpha)}\right]_0^\infty = \frac{1}{\alpha}$.

For $\int_0^\infty x\ e^{-\alpha x}dx = \left[x\ \frac{e^{-\alpha x}}{(-\alpha)}\right]_0^\infty - \int_0^\infty \frac{e^{-\alpha x}}{(-\alpha)}dx = \frac{1}{\alpha^2}$.

For $\int_0^\infty x^2\ e^{-\alpha x}dx = \left[x^2\ \frac{e^{-\alpha x}}{(-\alpha)}\right]_0^\infty - \int_0^\infty 2x\frac{e^{-\alpha x}}{(-\alpha)}dx = \frac{2}{\alpha^3}$.

For $\int_0^\infty x^3\ e^{-\alpha x}dx = \left[x^3\ \frac{e^{-\alpha x}}{(-\alpha)}\right]_0^\infty - \int_0^\infty 3x^2\frac{e^{-\alpha x}}{(-\alpha)}dx = \frac{3.2}{\alpha^4}$.

These results are seen to be in accord with the general result given.

We have successively $\dfrac{d}{d\alpha}\displaystyle\int_0^\infty e^{-\alpha x}dx = -\int_0^\infty x\ e^{-\alpha x}dx = \dfrac{d}{d\alpha}\dfrac{1}{\alpha} = -\dfrac{1}{\alpha^2}$ and then

$\dfrac{d}{d\alpha}\displaystyle\int_0^\infty x\ e^{-\alpha x}dx = -\int_0^\infty x^2\ e^{-\alpha x}dx = \dfrac{d}{d\alpha}\dfrac{1}{\alpha^2} = -\dfrac{2}{\alpha^3}$. Finally, in this question,

$$\frac{d}{d\alpha}\int_0^\infty x^2\ e^{-\alpha x}dx = -\int_0^\infty x^3\ e^{-\alpha x}dx = \frac{d}{d\alpha}\frac{2}{\alpha^3} = -\frac{3.2}{\alpha^4}\ .$$

4.

Imagine an extra thickness dx of ice forming over an area A of the ice water interface. The mass of ice forming is $A\ dx\ \rho$ kg and the heat that must be removed is $dQ = A\ dx\ \rho\ L$ where L is the latent heat of fusion of the ice. This element of heat is removed to the surface of the lake by thermal conduction. We have

$$\frac{dQ}{dt} = \frac{A\ dx\ \rho\ L}{dt} = \kappa\ A\ \frac{(T_s - 0)}{x} \quad \text{whence} \quad dt = \frac{\rho\ L}{\kappa\ T_s}x dx \quad \text{and} \quad t = \frac{1}{2}\frac{\rho\ L}{\kappa\ T_s}\ x^2\ .$$