1B45 Mathematical Methods Problem Class 1 2005/2006

Week starting Monday 24th. October

Solutions

1. We have $ax^2 + bx + c = x^2 + \frac{b}{a}x + \frac{c}{a} = (x + \frac{b}{2a})^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$

$$\text{so} \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \quad \text{and} \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

A quadratic may be factorized if $b^2 - 4ac$ is a perfect square.

If α and β are roots then

$$(x - \alpha)(x - \beta) = x^2 - (\alpha + \beta)x + \alpha\beta = 0 .$$

Combining this with the second equation on the first line we find

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a} .$$

a) For $2x^2 + x - 3 = 0$, $b^2 - 4ac = 25 = 5^2$,

$$2x^2 + x - 3 = 2x^2 + 3x - 2x - 3$$

$$= 2x^2 - 2x + 3x - 3 = 2x(x - 1) + 3(x - 1) = 0 .$$

$$\text{i.e.} \quad x = 1 \quad \text{or} \quad x = \frac{-3}{2} .$$

b) For $2x^2 + 7x + 3 = 0$, $b^2 - 4ac = 25 = 5^2$,

$$2x^2 + 7x + 3 = 2x^2 + 6x + x + 3$$

$$= 2x^2 + x + 6x + 3 = x(2x + 1) + 3(2x + 1) = 0 ,$$

$$\text{i.e.} \quad x = -\frac{1}{2} \quad \text{or} \quad x = -3 .$$

c) For $2x^2 - 9x + 5 = 0$, $b^2 - 4ac = 41$

$$\text{i.e.} \quad x^2 - \frac{9}{2}x + \frac{5}{2} = \left(x - \frac{9}{4}\right)^2 - \frac{81}{16} + \frac{5}{2}$$

$$= \left(x - \frac{9}{4}\right)^2 - \frac{81}{16} + \frac{40}{16} = 0$$

$$\text{i.e.} \quad x = \frac{9 \pm \sqrt{41}}{4} .$$

2. a) Let

$$\frac{x^2+3}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{(x^2+2)} \ ,$$

thus $x^2+3 = A(x^2+2) + (Bx+C)x$.
Choosing $x=0$ we find $A = \frac{3}{2}$ and $x^2+3 = \frac{3}{2}x^2 + 3 + Bx^2 + Cx$.
Comparing the coefficients of x^2 , we find $B = -\frac{1}{2}$.
Now choose $x=0$ again to find that $C=0$.

$$\text{Hence} \quad \frac{x^2+3}{x(x^2+2)} = \frac{3}{2x} - \frac{x}{2(x^2+1)}$$

b) Let

$$\frac{3}{x(3x-1)^2} = \frac{A}{x} + \frac{B}{(3x-1)} + \frac{C}{(3x-1)^2}$$

or $3 = A(3x-1)^2 + Bx(3x-1) + Cx$.
Choosing $x=0$ we find $A=3$.
Choosing $x = \frac{1}{3}$ we find $C=9$.
Then setting $x=1$ we fine $B=-9$.

$$\text{Hence} \quad \frac{3}{x(3x-1)^2} = \frac{3}{x} - \frac{9}{(3x-1)} + \frac{9}{(3x-1)^2}$$

3. a) For a right angle triangle

$$a^2+b^2=c^2 \quad \text{and} \quad \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$

$$\text{and} \quad \cos^2\theta + \sin^2\theta = 1 \ .$$

Dividing by $\cos^2\theta$ we find

$$1 + \tan^2\theta = \sec^2\theta$$

and dividing by $\sin^2\theta$ we find

$$1 + \cot^2\theta = \csc^2\theta \ .$$

b) Since $\cos(A+B) = \cos A\cos B - \sin A \sin B$

$$\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$$

(since $\cos^2 A = 1 - \sin^2 A$) .
Since $\sin(A+B) = \sin A\cos B + \cos A \sin B$,

$$\sin 2A = 2\sin A\cos A \quad \text{and} \quad \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$$

$$= 2\tan\frac{A}{2}\cos^2\frac{A}{2} = \frac{2\tan\frac{A}{2}}{\sec^2\frac{A}{2}} = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}} \ .$$

$$\text{Since} \quad \cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2} = \cos^2\frac{A}{2}(1 - \tan^2\frac{A}{2}) = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}} \; .$$

c) For $2\cos^2\theta - \sin\theta = 1$ we have $2(1-\sin^2\theta) - \sin\theta = 1$ or $2\sin^2\theta + \sin\theta - 1$. This may be factorized as $(2\sin\theta - 1)(\sin\theta + 1) = 0$.

$$\text{Hence} \quad \sin\theta = \frac{1}{2} \quad \text{or} \quad -1 \quad \text{and} \quad \theta = \frac{\pi}{6}\,,\frac{5\pi}{6} \quad \text{or} \quad \frac{3\pi}{2} \; .$$

For $\cos 2\theta + 3\sin\theta = 2$ use $\cos 2\theta = 1 - 2\sin^2\theta$ to obtain $1 - 2\sin^2\theta + 3\sin\theta$.

$$\text{i.e.} \quad 2\sin^2\theta - 3\sin\theta + 1 = (2\sin\theta - 1)(\sin\theta - 1) = 0$$

$$\text{and} \quad \sin\theta = \frac{1}{2} \quad \text{or} \quad 1 \quad \text{i.e.} \quad \theta = \frac{\pi}{6}\,,\frac{5\pi}{6} \quad \text{or} \quad \frac{\pi}{2} \; .$$

For $\sin\theta + 2\cos\theta = 1$ use the expression for $\sin\theta$ and $\cos\theta$ in terms of $\tan\frac{\theta}{2}$. Let $\tan\frac{\theta}{2} = t$. Then

$$\frac{2t}{1+t^2} + \frac{2(1-t^2)}{(1+t^2)} = 1$$

which yields $3t^2 - 2t - 1 = (3t+1)(t-1) = 0$.

$$\text{So} \quad \tan\frac{\theta}{2} = -\frac{1}{3} \quad \text{or} \quad 1 \quad \text{and} \quad \frac{\theta}{2} = 161.57^\circ \quad \text{or} \quad \frac{\theta}{2} = 45^\circ \; ,$$

$$\text{i.e.} \quad \theta = 323.14^\circ \quad \text{or} \quad 90^\circ \; .$$

(Note - substituting for *sine* in the above would involve a square root and a loss of sign information on squaring.)

4. We have

$$\epsilon_1 = 1 - \frac{T_{C1}}{T_H} \quad \text{and} \quad \epsilon_2 = 1 - \frac{T_{C2}}{T_H}$$

$$\Delta\epsilon = \epsilon_2 - \epsilon_1 = \frac{T_{C1} - T_{C2}}{T_H}$$

which is negative if $T_{C2} > T_{C1}$

$$\frac{\Delta\epsilon}{\epsilon_1} = \frac{T_{C1} - T_{C2}}{T_H}\frac{T_H}{T_H - T_{C1}} = \frac{T_{C1} - T_{C2}}{T_H - T_{C1}}$$

For a gas power plant $\frac{\Delta\epsilon}{\epsilon_1} = \frac{-15}{385} = 0.038$ i.e. 3.8%.
For a PWR nuclear plant $\frac{\Delta\epsilon}{\epsilon_1} = \frac{-15}{235} = 0.064$ i.e. 6.4%.

5. Sound takes time to travel from the car to the observer, Thus if the observer receives the pulse at time t, then the (retarded) time this signal is emitted is t minus the time the sound takes to cover the distance between the retarded position of the car to the observer.

$$\text{Thus} \quad [t] = t - \frac{[|\overrightarrow{r}|]}{c_s} \quad \text{where} \quad [|\overrightarrow{r}|] = \sqrt{(z - v[t])^2 + x^2 + y^2}$$

(where $z - v[t]$ is the retarded z position of car)

$$\text{i.e.} \quad c[t] - ct = [\overrightarrow{r}]$$

and squaring

$$c^2[t] - 2ct[t] + c^2t^2 = z^2 + v[t]^2 - 2zv[t] + x^2 + y^2$$

or

$$(c^2 - v^2)[t]^2 + 2(zv - c^2t)[t] - (x^2 + y^{2)} - z^2 + c^2t^2 = 0$$

and

$$[t]^2 + \frac{2(zv - c^2t)}{(c^2 - v^2)}[t] - \frac{(x^2 + y^2) + z^2 - c^2t^2}{(c^2 - v^2)} = 0$$

The above is a quadratic in $[t]$, which we now solve by completing the square.

We find

$$\left[[t] + \frac{(vz - c^2t)}{c^2 - v^2}\right]^2 = \frac{(vz - c^2t)^2 + (c^2 - v^2)(z^2 + (x^2 + y^2) - c^2t^2)}{(c^2 - v^2)^2}$$

The above, taking the negative square to ensure retardation and a very steady nerve yields

$$[t] = \frac{t - \frac{vz}{c^2} - \frac{1}{c}\sqrt{(z - vt)^2 + (1 - \frac{v^2}{c^2})(x^2 + y^2)}}{(1 - \frac{v^2}{c^2})}$$