1B45 Mathematical Methods Problem Sheet 10 Solutions 2005/2006

1.

(a) The magnitude of the position vector is a constant k.
Thus the position vector **ends on a sphere of radius** k. **[1]**

(b) The position vector 'dotted' with the unit vector $\hat{u}$ equals l.
Hence the position vector ends on a **plane where $\hat{u}$ is perpendicular to the plane and l is the perpendicular distance from the plane to the origin.** **[1]**
(This is the standard equation of the plane given in the lectures.)

(c) The position vector 'dotted' with the unit vector $\hat{u}$ equals m times the magnitude of $\vec{r}$ where $-1 \leq m \leq +1$.
Thus the position vector **lies in the surface of a cone with axis $\hat{u}$ and the cosine of the cone angle is m.** **[1]**

(d) The position vector **ends on the surface of a cylinder whose axis is given by $\hat{u}$ and is of radius n.** **[1]**

If the position vectors are $\vec{A} = 3\hat{i} - 4\hat{j} + 0\hat{k}$ and $\vec{B} = -2\hat{i} + 1\hat{j} + 0\hat{k}$ then $|A| = \sqrt{9+16+0} = 5$ and $|B| = \sqrt{4+1+0} = \sqrt{5}$. Thus

$$|A||B|cos\theta = 5\sqrt{5} = A_xB_x + A_yB_y + A_zB_z = -10 \quad \text{whence} \quad cos\theta = -\frac{2}{5} \quad \text{and} \quad \theta = 153.4^0.$$

[3]

$$\text{Now } \left(\vec{A} \times \vec{B}\right) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & 0 \\ -2 & 1 & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} - 5\hat{k}$$

This vector only has a component in the z direction and the unit vector can be taken as $(0,0,1)$, and these components are also the required direction cosines. **[3]**

From the standard form of the equation for a plane we can conclude that both planes go through the origin (the rhs of both equations are zero), and $(1,2,3)$ is a vector perpendicular to the first, and $(3,2,1)$ perpendicular to the second, plane.

$$\text{Thus } \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = -4\hat{i} + 8\hat{j} - 4\hat{k} \text{ determines the direction of the line of intersection.}$$

Normalising this vector we get $(-1,2,-1)\sqrt{6}$ and these are also the required direction cosines. **[3]**

We already have one point common to the two planes , namely the origin $(0,0,0)$. The two equations of the planes, when solved simultaneously determine two coordinates of the common point if one of the coordinates is given or chosen. For example if we multiply the second equation by 3 and eliminate z we find $x = -y/2$. So if we chose $y = 1$ then $x = -1/2$ and $z = -1/2$. Since the origin is a common point as well then the second common position vector is also the line of intersection and it is indeed the same vector found from the vector product above. **[2]**

2.

(a)

We have $e^{2iz} = e^{2i(x+iy)} = e^{-2y}e^{2ix} = e^{-2y}(cos2x{+}isin2x)$ and $Re\, e^{2iz} = e^{-2y}cos2x$.

[2]

(b) On the Argand diagram, which is the same as a polar plot except that the y axis is imaginary, the complex number $z = -1 + i\sqrt{3}$ is in the second quadrant since on the Argand diagram $x = -1$ and $iy = i\sqrt{3}$. The modulus is $\sqrt{1^2 + \sqrt{3}^2}$ ie 2 and in the second quadrant the angle α is given by $\alpha = cos^{-1}\sqrt{3}/2$ ie 60^0. Thus the polar angle, measured anti clockwise from the positive x axis is 120^0 or $2\pi/3$.

Thus $-1 + i\sqrt{3} = 2e^{i\frac{2\pi}{3}} = 2e^{i\frac{2\pi}{3}} \times 1 = 2e^{i\frac{2\pi}{3}}e^{2\pi n}$ and $(-1 + i\sqrt{3})^{\frac{1}{2}} = 2^{\frac{1}{2}}e^{\frac{1}{2}(i\frac{2\pi}{3}+2\pi n)}$

where $n = 0, 1$. For $n = 0$ $\quad (-1 + i\sqrt{3})^{\frac{1}{2}} = 2^{\frac{1}{2}}e^{\frac{i\pi}{3}} = \frac{1}{\sqrt{2}} + i\frac{\sqrt{3}}{\sqrt{2}}$

and for $n = 1$ $\quad (-1 + i\sqrt{3})^{\frac{1}{2}} = 2^{\frac{1}{2}}e^{\frac{i4\pi}{3}} = -\frac{1}{\sqrt{2}} - i\frac{\sqrt{3}}{\sqrt{2}}$

Note that higher values of n just repeat the above roots. **[4]**

(c)

$$\text{Now } i = e^{\frac{i\pi}{2}} \times 1 = e^{i(\frac{\pi}{2}+2n\pi)} \;,\quad n = 0 \text{ gives } i^{\frac{1}{2}} = e^{\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}} \;;$$

$$\text{and } \quad n = 1 \text{ gives } i^{\frac{1}{2}} = e^{\frac{3\pi}{4}} = \frac{-1-i}{\sqrt{2}} \;.$$

For $n = 0$ $|z| = \sqrt{zz^*} = \left(e^{\frac{1+i}{\sqrt{2}}}e^{\frac{1-i}{\sqrt{2}}}\right)^{\frac{1}{2}} = e^{\frac{1}{\sqrt{2}}}$ and for $n = 1$ $|z| = \left(e^{\frac{-1-i}{\sqrt{2}}}e^{\frac{-1+i}{\sqrt{2}}}\right)^{\frac{1}{2}} = e^{-\frac{1}{\sqrt{2}}}$.

[4]

(d)

Since we have already seen that $i = e^{i(\frac{\pi}{2}+2\pi n)}$, $i^i = e^{i(\frac{\pi}{2}+2\pi n)i} = e^{(-\frac{\pi}{2}-2\pi n)}$.

[2]

(e)

Since $B_0 = \sqrt{i}C\ A_0 = (1 + i)/\sqrt{2}\ C\ A_0$ we can conclude that B leads A by a phase shift of 45^0 by plotting the amplitudes on an Argand diagram. **[3]**