1B45 Mathematical Methods Problem Sheet 9 Solutions 2005/2006

1. For the first line we have

$$2x + y + 3z = 1 \quad \text{and} \quad x + 10y + 0z = 21 \; .$$

Solving the above simultaneously, eliminating x we find

$$19y - 3z - 41 = 0 \quad \text{or} \quad 19(y-2) - 3(z+1) = 0 \quad \text{or} \quad \frac{z+1}{19} = \frac{y-2}{3} \; .$$

Eliminating y we find

$$19x + 30z + 11 = 0 \quad \text{or} \quad 19(x-1) + 30(z+1) = 0 \quad \text{or} \quad \frac{z+1}{19} = \frac{(x-1)}{-30} \; .$$

So for the first line

$$\frac{(x-1)}{-30} = \frac{(y-2)}{3} = \frac{z+1}{19}$$

and the vector equation for the line is

$$\vec{r_1} = (\hat{i} + 2\hat{j} + \hat{k}) + (-30\hat{i} + 3\hat{j} + 19\hat{k})\lambda_1$$

By inspection the second line can almost be directly written down

$$\frac{x}{1} = \frac{y}{2} = \frac{z-6}{-7} \quad \text{and} \quad \vec{r_2} = (0\hat{i} + 0\hat{j} + 6\hat{k}) + (\hat{i} + 2\hat{j} - 7\hat{k})\lambda_2$$

For the lines to intersect the minimum distance between them must be zero. ie

$$\left|\vec{d}\right| = (\vec{a_2} - \vec{a_1}) \cdot \left(\hat{b_2} \times \hat{b_1}\right) = 0. \text{ ie } \vec{a_2} \cdot \left(\hat{b_2} \times \hat{b_1}\right) = \vec{a_1} \cdot \left(\hat{b_2} \times \hat{b_1}\right)$$

$$\text{or} \quad \vec{a_2} \cdot \left(\vec{b_2} \times \vec{b_1}\right) = \vec{a_1} \cdot \left(\vec{b_2} \times \vec{b_1}\right) \; .$$

$$\text{Now} \; \left(\vec{b_1} \times \vec{b_2}\right) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -7 \\ -30 & 3 & 19 \end{vmatrix} = 17\hat{i} + -229\hat{j} + 63\hat{k}$$

$$= (\hat{i} + 2\hat{j} - 1\hat{k}) \cdot (17\hat{i} - 229\hat{j} + 63\hat{k}) = 378 = (0\hat{i} + 0\hat{j} + 6\hat{k}) \cdot (17\hat{i} - 229\hat{j} + 63\hat{k}) = 378$$

Thus the lines interesect.

Point of intersection - it is easiest to solve the original component equations simultaneously.

For the first line we have $x + 10y = 20$ and for the second $2x = y$. Thus $y = 2$ and $x = 1$. We also have for the second line that $7x + z = 6$ so $z = -1$.

The coordinates of the intersection point is thus $(1, 2, -1)$.

For the equation of the plane we use

$$(\overrightarrow{r} - \overrightarrow{a}) \cdot \overrightarrow{N}$$

where $\overrightarrow{a}$ is a position vector on this plane, and $\overrightarrow{N}$ is a vector perpendicular on the plane.

We have $\vec{a}$ already ie

$$\overrightarrow{a} = \hat{\imath}+2\hat{\jmath}-\hat{k} \quad \text{and} \quad \overrightarrow{N} = 17\hat{\imath}-229\hat{\jmath}+63\hat{k}. \quad \text{Thus} \quad (\overrightarrow{r}-(\hat{\imath}+2\hat{\jmath}-\hat{k}))\cdot(17\hat{\imath}-229\hat{\jmath}+63\hat{k}) = 0$$

is an equation for the plane.

2. Since the line goes through the origin and the point $(2, 2, 5)$ we can write its vector equation as

$$\overrightarrow{r_1} = 0\hat{\imath} + 0\hat{\jmath} + 0\hat{k} + \lambda(2\hat{\imath} + 2\hat{\jmath} + 5\hat{k}) = \lambda(2\hat{\imath} + 2\hat{\jmath} + 5\hat{k}).$$

The position vector of the point $P_2(1, 2, 1)$ is given by

$$\overrightarrow{r_2} = \hat{\imath} + 2\hat{\jmath} + \hat{k} \ .$$

Suppose the foot of the perpendicular is at N then

$$\overrightarrow{P_2N} = \overrightarrow{r_{1N}} - \overrightarrow{r_2} = (2\hat{\imath} + 2\hat{\jmath} + 5\hat{k})\lambda_N - (\hat{\imath} + 2\hat{\jmath} + \hat{k})$$

where λ_N is the value of λ at the foot of the perpendicular.

But $\overrightarrow{P_2N}$ is perpendicular to the line $\overrightarrow{r_1}$. Therefore $(\overrightarrow{P_2N} \cdot \overrightarrow{b_1}) = 0$ ie

$$((2\hat{\imath} + 2\hat{\jmath} + 5\hat{k})\lambda_N - (\hat{\imath} + 2\hat{\jmath} + \hat{k})) \cdot (2\hat{\imath} + 2\hat{\jmath} + 5\hat{k}) = 0 \ \text{ or } \ 33\lambda_N = 11 \ \text{ ie } \ \lambda_N = \frac{1}{3} \ .$$

Putting $\lambda_N = \frac{1}{3}$ into the equation for $\overrightarrow{r_1}$ we find the coordinates to be $(\frac{2}{3}, \frac{2}{3}, \frac{5}{3})$.

3. The Maclaurin series is $f(x) = f(0) + \frac{df(0)}{dx}x + \frac{1}{2!}\frac{d^2f}{dx^2}x^2 + \frac{1}{3!}\frac{d^3f}{dx^3}x^3 \ldots \frac{1}{n!}\frac{d^nf}{dx^n}x^n$. For $f(x) = e^x$, $f(0) = 1$ and $\frac{d^nf(0)}{dx^n} = 1$. Thus

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n!} \quad \text{and the sum would be}$$

$$S_n = \sum_{n=0}^{n} \frac{x^n}{n!} \; .$$

For $\cos x, f(0) = 1, \dfrac{df}{dx} = -\sin x \, , \dfrac{d^2f}{dx^2} = -\cos x, \dfrac{d^3f}{dx^3} = +\sin x,$ and $\dfrac{d^4f}{dx^4} = +\cos x$.

$$\text{Thus} \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - (-1)^n \frac{x^{2n}}{(2n)!} \; .$$

For $\sin x$ the easiest thing to do is to differentiate the series for $\cos x$. ie

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots (-1)^n \frac{x^{2n+1}}{(2n+1)!} \; .$$

For $f(x) = \ln(1+x)$ strictly speaking we are using the Taylor expansion for this ie $f(x) = f(1+x)$

We find $f(1) = \ln 1 = 0, \dfrac{df}{dx} = (1{+}x)^{-1}, \dfrac{d^2f}{dx^2} = (-1)(1{+}x)^{-2}, \dfrac{d^3f}{dx^3} = (-1)(-2)(1{+}x)^{-3}$,

$$\frac{d^4f}{dx^4} = (-1)(-2)(-3)(1+x)^{-4} \text{ and so on .}$$

$$\text{Thus} \quad \ln(1+x) = x - \frac{x^2}{2!} + \frac{2!}{3!}x^3 + \frac{3!}{4!}x^4 + \cdots + (-1)^{n+1}\frac{x^n}{n}$$

For $f(x) = \ln\left(\frac{1+x}{1-x}\right)$ write $f(x) = \ln(1+x)$.
Then

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots - (x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \cdots)$$

$$= 2(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} \cdots)$$

The general term here for the sum is

$$S_n = \sum_{n \text{ odd}} 2\frac{x^n}{n} \; .$$