1B45 Mathematical Methods Problem Sheet 7 Solutions 2005/2006

1.

Taylor Series Expansion in two dimensions. (First part directly from notes.)

Here we suppose we know the function at $(x, y) = (a, b)$ and we want to predict its value at x, y a small distance $(x - a)$ and $(y - b)$ away. We assume that $f(x, y)$ may be written as a series of terms in powers of $(x - a)$ and $(y - b)$. ie

$$f(x,y) = a_{00} + a_{10}(x-a) + a_{01}(y-b) + a_{20}(x-a)^2 + a_{02}(y-b)^2 + a_{11}(x-a)(y-b)...$$

Putting $x = a$ and $y = b$ in the above we find $a_{00} = f(a, b)$.

$$\text{Now } \left(\frac{\partial f(x,y)}{\partial x}\right)_y = 0 + a_{10} + 0 + 2a_{20}(x-a) + 0 + a_{11}(y-b)...$$

Putting $y = b$ and $x = a$ we find $a_{10} = \left(\frac{\partial f(a,b)}{\partial x}\right)_y$

$$\text{Now } \frac{\partial f(x,y)}{\partial y} = 0 + 0 + a_{01} + 0 + 2a_{02}(y-b) + a_{11}(x-a)...$$

Putting $x = a$ and $y = b$, we find $a_{01} = \left(\frac{\partial f(a,b)}{\partial y}\right)_x$

$$\text{We also find that } \frac{\partial^2 f(a,b)}{\partial x^2} = 2a_{20}, \ \frac{\partial^2 f(a,b)}{\partial y^2} = 2a_{02} \text{ and } \frac{\partial^2 f}{\partial x \partial y} = a_{11} \ .$$

$$\text{Thus } f(x,y) = f(a,b) + \left(\frac{\partial f(a,b)}{\partial x}\right)_y (x-a) + \left(\frac{\partial f(a,b)}{\partial y}\right)_x (y-b)$$

$$+\frac{1}{2}\left\{\frac{\partial^2 f(a,b)}{\partial x^2}(x-a)^2 + 2\frac{\partial^2 f(a,b)}{\partial x \partial y}(x-a)(y-b) + \frac{\partial^2 f(a,b)}{\partial y^2}(y-b)^2\right\} \ .$$

[2]

The term involving the second derivatives can be written by just completing the square, ie

$$\frac{1}{2}\left(f_{xx}\Delta x^2 + 2f_{xy}\Delta x\Delta y + f_{yy}\Delta y^2\right) = \frac{1}{2}\left\{f_{xx}\left(\Delta x + \frac{f_{xy}}{f_{xx}}\Delta y\right)^2 + \Delta y^2\left(f_{yy} - \frac{f_{xy}^2}{f_{xx}}\right)\right\} \ .$$

[2]

For

$$f(x,y) = x^3 + xy^2 - 12x - y^2 \ \text{ we find } \ \frac{\partial f}{\partial x} = 3x^2 + y^2 - 12 \ \text{ and } \ \frac{\partial f}{\partial y} = 2xy - 2y \ .$$

From the last equation we find $y = 0$ and $x = 1$ and from the penultimate equation,equating them both to zero, we then find the following coordinates of stationary points $(+2, 0)$, $(-2, 0)$, $(1, +3)$ and $(1, -3)$. **[3]**

The second derivatives are given by

$$f_{xx} = 6x \ \ , \ f_{yy} = 2x - 2 \ \text{ and } \ f_{xy} = 2y \ .$$

For the coordinate $(-2, 0)$, $f_{xx} = -12$, $f_{yy} = -6$ and $f_{xy} = 0$. Since f_{xx} and f_{yy} are both negative, and $f_{xx}f_{yy} > f_{xy}^2$ there is a maximum at this point. (The value of $f(x, y)$ is 16 - not actually requested.) **[3]**

2.
The mass flow of gas in a nozzle is given by

$$\frac{|\vec{v_2}|}{\nu_2} = \left[\frac{2p_1}{\nu_1}\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}}\left(1-\left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\gamma}}\right)\right]^{\frac{1}{2}} .$$

The term that determines the maximum is simply

$$\left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}}\left(1-\left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\gamma}}\right) \quad \text{or, multiplying out and substituting,} \quad x^{\frac{2}{\gamma}} - x^{1+\frac{1}{\gamma}} .$$

Differentiating and equating to zero **[3]**

$$\frac{2}{\gamma}x^{\frac{2}{\gamma}-1} - (1+\frac{1}{\gamma})x^{\frac{1}{\gamma}} = x^{\frac{1}{\gamma}}\left(\frac{2}{\gamma}x^{\frac{1}{\gamma}-1} - (1+\frac{1}{\gamma}\right) = 0$$

$$\text{ie } x^{\frac{1-\gamma}{\gamma}} = \frac{\gamma+1}{2} \quad \text{or} \quad x^{\frac{\gamma-1}{\gamma}} = \frac{2}{\gamma+1} \quad \text{and} \quad x = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$$

We have at maximum flow **[4]**

$$T_2 = T_1\frac{p_2}{p_1}\frac{\nu_2}{\nu_1} = T_1\frac{p_2}{p_1}\left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}} = T_1\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = T_1\left(\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}\right)^{\frac{\gamma-1}{\gamma}} = T_1\frac{2}{\gamma+1} .$$

[3]

3.
From the van der Waals equation we have

$$p = \frac{RT}{(V-b)} - \frac{a}{V^2} \quad \text{from which} \quad \left(\frac{\partial p}{\partial V}\right)_T = (-1)RT(V-b)^{-2} + 2aV^{-3} .$$

[2]

Taking differentials of everything in sight starting with

$$p(V-b) + \frac{a}{V} - \frac{ab}{V^2} \quad \text{we get} \quad dp(V-b) + pdV - \frac{a}{V^2}dV + 2\frac{ab}{V^3}dV = RdT .$$

Setting $dT = 0$ we get

$$\left(dp(V-b) = (-p + \frac{a}{V^2} - 2\frac{ab}{V^3})dV\right)_T \quad \text{ie} \quad \left(\frac{\partial p}{\partial V}\right)_T = \frac{1}{(V-b)}\left(-\frac{RT}{(V-b)} + 2\frac{a}{V^2} - 2\frac{ab}{V^3}\right)$$

Taking the factor $(V-b)$ out of the last two terms yields the same result as above, albeit with more trouble. **[3]**

Setting $dV = 0$ in the differential expression, or directly from the first line

$$\left(\frac{\partial p}{\partial T}\right)_V = \frac{R}{V-b} .$$

[2]

From $\left(\frac{\partial p}{\partial V}\right)_T = -RT(V-b)^{-2} + 2aV^{-3}$ we find $\left(\frac{\partial^2 p}{\partial V^2}\right)_T = 2RT(V-b)^{-3} + 6aV^{-4}$

Setting these derivatives to zero we obtain

$V_c^3RT_c = 2(V_c-b)^2a$ and $V_c^4RT_c = 3(V_c-b)^3a$. Dividing we find $V_c = 3b$.

Substituting this in the first equation yields $T_c = \frac{8}{27}\frac{a}{b}\frac{1}{R}$.

[3]