1B45 Mathematical Methods Problem Sheet 5 Solutions 2005/2006

1. From the compound angle expression we have

$$\sin 2x = 2\sin x\cos x \quad \text{or} \quad \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\tan\frac{x}{2}/\sec^2\frac{x}{2} = \frac{2t}{1+t^2}\ .$$

[3]

$$\text{If}\quad t=\tan\frac{x}{2},\ dt = \frac{1}{2}\sec^2\frac{x}{2}\ dx = \frac{(1+t^2)}{2}dx \quad \text{whence} \quad dx = \frac{2\ dt}{1+t^2}\ .$$

[3]

$$\text{Thus}\quad \int \csc x\ dx = \int\frac{dx}{\sin x} = \int\frac{(1+t^2)}{2t}\frac{2dt}{(1+t^2)} = \ln t = \ln\tan\frac{x}{2}\ .$$

[4]

2. We have

$$\left(M+m(1-\frac{t}{T})\right)\frac{dv}{dt} = F \quad \text{or} \quad \frac{dt}{M+m\left(1-\frac{t}{T}\right)} = \frac{dv}{F}\ .$$

[3]

$$\text{Thus}\quad \frac{-T}{m}\ln\left(M+m\left(1-\frac{t}{T}\right)\right) = \frac{v}{F} + \text{const}$$

[2]

$$\text{or}\quad v_f - v_i = \left[\frac{-FT}{m}\ln\left(M+m\left(1-\frac{t}{T}\right)\right)\right]_{t_i}^{t_f}\ .$$

[2]

$$\text{When}\quad t_i=0,\ \ v_i=0\ \ \text{so}$$

$$v_f = \frac{FT}{m}\left[-\ln M + \ln(M+m)\right] = \frac{FT}{m}\ln\left(\frac{M+m}{M}\right)\ .$$

[3]

3. The curve $e^{-\alpha x^2}$ can be rotated about the z axis, creating a solid of revolution whose volume equals I^2, where $I = \int_{-\infty}^{+\infty} e^{-\alpha x^2}dx = \int_{-\infty}^{+\infty} e^{-\alpha y^2}dy$. This volume is calculated by slicing the volume of revolution into circular slices of area $\pi r^2 = \pi(x^2+y^2)$ and thickness dz.

$$\text{Then}\quad V = \int_0^1 \pi r^2 dz = -\int_0^1 \frac{\pi}{2}\ln z\ dz$$

$$= \frac{\pi}{2}\left[z(\ln z - 1)\right]_1^0 \quad \text{since}\quad z = e^{-\alpha(x^2+y^2)} = e^{-\alpha r^2}\ .$$

Since $z \ln z = 0,\ z \to 0$

$$V = I^2 = \frac{\pi}{\alpha},\ \ I = \int_{-\infty}^{+\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}}$$

$$\text{and} \quad \int_0^{\infty} e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \ .$$

[4]

Differentiating both sides of the above with respect to α we get

$$\frac{d}{d\alpha}\int_0^{\infty} e^{-\alpha x^2} dx = \int_0^{\infty} -x^2 e^{-\alpha x^2} dx$$

$$= \frac{d}{d\alpha}\frac{\sqrt{\pi}}{2}\alpha^{-\frac{1}{2}} = \frac{1}{2}\left(-\frac{1}{2}\right)\sqrt{\pi}\alpha^{-\frac{3}{2}} \quad \text{or} \quad \int_0^{\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{4}\sqrt{\frac{\pi}{\alpha^3}} \ .$$

[3]

We require

$$\int_0^{\infty} P(v) dv = 4\pi N \int_0^{\infty} v^2 e^{-\frac{1}{2}\frac{mv^2}{kT}} dv = 1 \ ,$$

$$\text{ie} \quad N 4\pi \frac{\sqrt{\pi}}{4}\left(\frac{2kT}{m}\right)^{\frac{3}{2}} = 1 \ .$$

$$\text{Thus} \quad N = \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \ .$$

[3]