1B45 Mathematical Methods Problem Sheet 4 Solutions 2005/2006

1. We have $\cosh x = \frac{e^x+e^{-x}}{2}$ and $\sinh x = \frac{e^x-e^{-x}}{2}$.
Adding, $\cosh x + \sinh x = e^x$.
Squaring and subtracting we find

$$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{2x} + 2}{4} = 1 \ .$$

If $y = \cosh^{-1}\frac{x}{a}$ then $\frac{x}{a} = \cosh y$.
Now $e^y = \cosh y + \sinh y = \cosh y \pm \sqrt{\cosh^2 y - 1}$. So

$$y = \ln e^y = \ln\left(\frac{x}{a} \pm \sqrt{\frac{x^2}{a^2} - 1}\right) \ .$$

i.e.

$$y = \cosh^{-1}\left(\frac{x}{a}\right) = \ln\left(\frac{x \pm \sqrt{x^2 - a^2}}{a}\right) \ .$$

Now $\frac{x-\sqrt{x^2-a^2}}{a} = \frac{a}{\left(x+\sqrt{x^2-a^2}\right)}$ as can be verified by cross multiplication.
So $\ln\left((x - \sqrt{x^2 - a^2})/a)\right) = -\ln\left((x + \sqrt{x^2 - a^2})/a\right)$. Hence $y = \pm\ln\left((x + \sqrt{x^2 - a^2})/a\right)$.
If $y = \sinh^{-1}\frac{x}{a}$ then $\frac{x}{a} = \sinh y$.
As before $e^y = \cosh y + \sinh y = \sqrt{1 + \sinh^2 y} + \sinh y$. So

$$y = \ln\left(\frac{x}{a} \pm \sqrt{1 + \frac{x^2}{a^2}}\right)$$

$$= \ln\left(\frac{x + \sqrt{a^2 + x^2}}{a}\right) \ .$$

where we have taken the positive square root because cosh is never negative.
If $y = \tanh^{-1}\frac{x}{a}$, $\frac{x}{a} = \tanh y = \frac{\sinh y}{\cosh y}$. So

$$\frac{x}{a} = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1} \ .$$

Thus $xe^{2y} + x = ae^{2y} - a$ and $e^{2y}(x - a) = -x - a$ or $e^{2y} = \frac{a+x}{a-x}$. Thus $y = \frac{1}{2}\ln\frac{a+x}{a-x}$.

2.a)

$$\frac{d}{dx}\left(x^2e^x\right) = 2xe^x + x^2e^x = xe^x(x+2)\ .$$

b)

$$\frac{d}{dx}\ln\left(a^x + a^{-x}\right) = \frac{1}{(a^x + a^{-x})}\left\{a^x \ln a + a^{-x}(-1)\ln a\right\}$$

$$= \ln a\frac{a^x - a^{-x}}{a^x + a^{-x}}\ .$$

c)

$$\frac{d}{dx}\ln\left(x^a + x^{-a}\right) = \frac{1}{x^a + x^{-a}}\left\{ax^{a-1} + (-a)x^{-a-1}\right\}$$

$$= \frac{\left(ax^{(a-1)} - ax^{-a-1}\right)}{x^a + x^{-a}} = \frac{a\left(x^a - x^{-a}\right)}{x\left(x^a + x^{-a}\right)}\ .$$

d) If $y = x^x$ then $\ln y = x\ln x$

$$\text{i.e.}\quad \frac{dy}{y} = dx\ln x + x\frac{dx}{x}$$

$$\text{Thus}\quad \frac{dy}{dx} = y\left(\ln x + 1\right)$$

$$\text{and}\quad \frac{dy}{dx} = x^x(\ln x + 1)\ .$$

e)

$$\frac{d}{dr}\left(r^2 + d^2 - 2rd\cos\theta\right)^{-\frac{1}{2}} = -\frac{1}{2}\frac{2r - 2d\cos\theta}{(r^2 + d^2 - 2rd\cos\theta)^{\frac{3}{2}}}$$

$$= \frac{r - d\cos\theta}{(r^2 + d^2 - 2rd\cos\theta)^{\frac{3}{2}}}\ .$$