1B45 Mathematical Methods Problem Sheet 3 Solutions 2005/2006

1. In $2^{2x}+3(2^x)-4=0$ substitute $y=2^x$ then

$$y^2+3y-4=(y+4)(y-1)=0\ ,$$

$$\text{i.e.}\quad y=-4 \quad\text{or}\quad y=1 \quad\text{and}\quad 2^x=-4 \quad\text{or}\quad 2^x=1\ .$$

There are no real values of x for which $2^x=-4$. But if $2^x=1$ then $x=0$ is the only solution to this equation. **[6]**

2. The quadratic is

$$E^2-(H_{11}+H_{22})E+H_{12}H_{21}=0\ .$$

Completing the square we find

$$\left(E-\frac{(H_{11}+H_{22})}{2}\right)^2-\frac{(H_{11}+H_{22})}{4}+H_{12}H_{21}=0 \quad\text{and}$$

$$E=\frac{(H_{11}+H_{22})}{2}\pm\sqrt{\frac{(H_{11}-H_{22})^2}{4}+H_{12}H_{21}}\ .$$

[5]

From the above

$$E=\frac{(H_{11}+H_{22})}{2}\pm\frac{(H_{11}-H_{22})}{2}\sqrt{1+\frac{4H_{12}H_{21}}{(H_{11}-H_{22})^2}}\ .$$

$$\approx\frac{(H_{11}+H_{22})}{2}\pm\frac{(H_{11}-H_{22})}{2}\left(1+\frac{1}{2}\frac{4H_{12}H_{21}}{(H_{11}-H_{22})^2}\quad+\ldots\ldots\right)\ .$$

Thus

$$E_+\approx H_{11}+\frac{H_{12}H_{21}}{(H_{11}-H_{22})}\quad\text{and}\quad E_-\approx H_{22}-\frac{H_{12}H_{21}}{(H_{11}-H_{22})}\ .$$

[5]

3. We have

$$u=v_1\cos\theta+\sqrt{2}v_2 \text{ and } v_1\sin\theta=\sqrt{2}v_2\ .$$

From the above

$$v_1=\frac{u}{\sin\theta+\cos\theta}\quad\text{and}\quad v_2=\frac{u\sin\theta}{\sqrt{2}(\sin\theta+\cos\theta)}\ .$$

Putting these into the third equation yields

$$u^2=\frac{u^2}{(\sin\theta+\cos\theta)^2}+\frac{2u^2\sin^2\theta}{2(\sin\theta+\cos\theta)^2}$$

$$\text{i.e.}\quad(\sin\theta+\cos\theta)^2=(1+\sin^2\theta)\ ,\qquad \sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta=1+\sin^2\theta\ ,$$

$$\text{i.e.}\quad 2\sin\theta\cos\theta=\sin^2\theta\ ,\qquad \sin^2\theta(1-2\cot\theta)=0 \quad\text{and}\qquad \sin\theta=0 \quad\text{or}\quad \tan\theta=2\ .$$

The value $\tan\theta=2$ corresponds to $\sin\theta=\frac{2}{\sqrt{5}}$ and $\theta=63.4°$. **[6]**

4. We have

$$|\vec{v_2}| = \left[\left(\frac{2\gamma}{\gamma-1}\right)(p_1\nu_1 - p_2\nu_2)\right]^{\frac{1}{2}}$$

$$= \left[\left(\frac{2\gamma}{\gamma-1}\right)p_1\nu_1\left(1 - \frac{p_2}{p_1}\frac{\nu_2}{\nu_1}\right)\right]^{\frac{1}{2}}$$

$$= \left[\left(\frac{2\gamma}{\gamma-1}\right)p_1\nu_1\left(1 - \frac{p_2}{p_1}\left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}}\right)\right]^{\frac{1}{2}}$$

$$= \left[\left(\frac{2\gamma}{\gamma-1}\right)p_1\nu_1\left(1 - \frac{p_2}{p_1}\left(\frac{p_2}{p_1}\right)^{-\frac{1}{\gamma}}\right)\right]^{\frac{1}{2}}$$

$$= \left[\left(\frac{2\gamma}{\gamma-1}\right)p_1\nu_1\left(1 - \left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\gamma}}\right)\right]^{\frac{1}{2}}$$

where we have used **[4]**

$$p_1\nu_1^\gamma = p_2\nu_2^\gamma \quad \text{or} \quad \frac{\nu_2}{\nu_1} = \left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}} .$$

Now

$$\frac{|\overrightarrow{\nu_2}|}{\nu_2} = \left[\frac{2\gamma}{\gamma-1}\frac{p_1\nu_1}{\nu_2^2}\left(1 - \left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\gamma}}\right)\right]^{\frac{1}{2}} .$$

Since $\nu_2 = \nu_1\left(\frac{p_1}{p_2}\right)^{\frac{1}{\gamma}}$

$$\frac{|\overrightarrow{\nu}_2|}{\nu_2} = \left[\frac{2\gamma}{\gamma-1}\frac{p_1\nu_1}{\nu_1^2}\left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}}\left(1 - \left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\gamma}}\right)\right]^{\frac{1}{2}}$$

$$= \left[\frac{2p_1}{\nu_1}\frac{\gamma}{\gamma-1}\left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}}\left(1 - \left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\gamma}}\right)\right]^{\frac{1}{2}} .$$

[4]