1B45 Mathematical Methods Problem Sheet 2 Solutions 2005/2006

1. Following procedures described in the lectures we have $x-a)^8$

$$= x^8 + \frac{8}{1}x^7(-a)^1 + \frac{8.7}{2.1}x^6(-a)^2 + \frac{8.7.6}{3.2.1}x^5(-a)^3 + \frac{8.7.6.5}{4.3.2.1}x^4(-a)^4$$

$$+\frac{8.7.6.5.4}{5.4.3.2.1}x^3(-a)^5 + \frac{8.7.6.5.4.3}{6.5.4.3.2.1}x^2(-a)^6 + \frac{8.7.6.5.4.3.2}{7.6.5.4.3.2.1}x(-a)^7 + (-a)^8$$

$$= x^8 - 8x^7a + 28x^6a^2 - 56x^5a^3 + 70x^4a^4 - 56x^3a^5 + 28x^2a^6 - 8xa^7 + a^8.$$

2. We have

$$S_{n+1} - S_n = (n+1)^3 = n^3 + 3n^2 + 3n + 1$$

Now assume that $S_n = \alpha n + \beta n^2 + \gamma n^3 + \delta n^4$, whence $S_{n+1} - S_n$

$$= \alpha(n+1) + \beta(n+1)^2 + \gamma(n+1)^3 + \delta(n+1)^4 - \alpha n - \beta n^2 - \gamma n^3 - \delta n^4$$

$$= \alpha + \beta(2n+1) + \gamma(3n^2+3n+1) + \delta(4n^3+6n^2+4n+1) = n^3 + 3n^2 + 3n + 1$$

where we have equated the two expressions for $S_{n+1} - S_n$.

Comparing coefficients of the powers of n we find $\delta = 1/4, \gamma = 1/2, \beta = 1/4$ and $\alpha = 0$ whence

$$S_n = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 = \frac{n^2(n^2+2n+1)}{4} = \left[\frac{n}{2}(n+1)\right]^2.$$

3. We have

$$S_n = \sum_{r=1}^{n}(r+1)(r+3) = \sum_{r=1}^{n}(r^2+4r+3) = \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} 4r + \sum_{r=1}^{n} 3$$

$$= \frac{1}{6}n(n+1)(2n+1) + 4 \times \frac{1}{2}n(n+1) + 3n = \frac{1}{6}n(2n^2+15n+31)$$

4. We have

$$S_n = a + (a+d)r + (a+2d)r^2 + (a+3d)r^3.....[a+(n-1)d]r^{n-1}$$

and multiplying by r, either because of mathematical genius, or applying the method of error and trial,

$$rS_n = \qquad ar + (a+d)r^2 + (a+2d)r^3................[a+(n-1)d]r^n$$

Subtracting the above we get

$$(1-r)S_n = a + rd + r^2d + r^3d... + r^{n-1}d - [a+(n-1)d]r^n = a - [a+(n-1)d]r^n + \frac{rd - r^nd}{1-r}$$

where we have summed the geometric series $rd + r^2d........r^{n-1}d$.

The final result is therefore

$$S_n = \frac{a - [a+(n-1)d]r^n}{1-r} + \frac{rd(1-r^{n-1})}{(1-r)^2}.$$