1B45 Mathematical Methods Problem Sheet 7 2005/2006

Staple securely your answer sheets together and put **your name** and your **tutor's name** (Prof. T. W. Jones if you are not in the P+A department) on your script.

Please hand in your solutions at the Monday Lecture 28th. November.

Revision of the Taylor expansion and stationary points in two variables.

1.

Obtain the Taylor expansions for a two variable function up to and including the second partial derivatives and show that

$$\left(f_{xx}\Delta x^2 + 2f_{xy}\Delta x\Delta y + f_{yy}\Delta y^2\right) = f_{xx}\left(\Delta x + \frac{f_{xy}\Delta y}{f_{xx}}\right)^2 + \Delta y^2\left(f_{yy} - \frac{f_{xy}^2}{f_{xx}}\right) .$$

Find the stationary points of $f(x, y) = x^3 + xy^2 - 12x - y^2$.

Determine the nature of the function at $x, y = -2, 0$. **[10]**

A single variable problem involving finding a maximum, plus manipulation of powers and algebra.

2.

The mass flow of gas in a nozzle is given by

$$\frac{|\vec{v_2}|}{\nu_2} = \left[\frac{2p_1}{\nu_1}\left(\frac{\gamma}{\gamma-1}\right)\left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}}\left(1 - \left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\gamma}}\right)\right]^{\frac{1}{2}} .$$

If $x = \frac{p_2}{p_1}$ show that at maximum flow $x = \frac{p_2}{p_1} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}$.

Using $p_1\nu_1^\gamma = p_2\nu_2^\gamma$ and $T_2 = T_1\frac{p_2\nu_2}{p_1\nu_1}$ show that at maximum flow $T_2 = \frac{2}{\gamma+1}T_1$. **[10]**

Differentials, partial derivatives plus some algebra.

3.

The van der Waals equation

$$\left(p + \frac{a}{V^2}\right)(V - b) = RT$$

is a remarkably good, but not perfect, description of a real gas where a and b are constants, p is the pressure, V the volume and T the temperature of the gas.

First, obtain an equation for p and show that $\left(\frac{\partial p}{\partial V}\right)_T = -RT(V-b)^{-2} + 2aV^{-3}$.

Now obtain the same result by taking differentials of the van der Waals equation and show also that

$$\left(\frac{\partial p}{\partial T}\right)_V = \frac{R}{V-b}$$

At the critical point $\left(\frac{\partial p}{\partial V}\right)_T = \left(\frac{\partial^2 p}{\partial V^2}\right)_T = 0$. Show that $V_c = 3b$ and $T_c = \frac{8}{27}\frac{a}{b}\frac{1}{R}$. **[10]**