Answer FIVE questions.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of marks per sub-section of a question.

1. (a) If the general quadratic equation is of the form $ax^2 + bx + c = 0$ show by 'completing the square' that

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \; .$$ **[4]**

(b) If the roots are α and β show by any method that

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a} \; .$$ **[4]**

(c) The energies E of a two state coupled quantum system are given by the equation

$$(E - H_{11})(E - H_{22}) - H_{12}H_{21} = 0 \; .$$

Show that the solutions for E are given by

$$E = \frac{H_{11} + H_{22}}{2} \pm \sqrt{\frac{(H_{11} - H_{22})^2}{4} + H_{12}H_{21}} \; .$$ **[6]**

(d) If $r = \frac{4H_{12}H_{21}}{(H_{11}-H_{22})^2} << 1$ determine the solutions for E to first order in r. **[6]**

2. (a) Write down the definition of the derivative $\frac{df(x)}{dx}$ in terms of a limiting procedure. **[4]**

(b) Apply the above definition to the function $y = a^x$, where a is a constant and explain carefully how the exponential function is defined by the following expression

$$y = \frac{dy}{dx} = e^x.$$ **[4]**

(c) How do the above considerations in (b) allow one to determine $\int \frac{dy}{y}$? **[3]**

(d) The angular frequency distribution of black body radiation $I(\omega)d\omega$ is given by

$$I(\omega)d\omega = \frac{1}{\pi^2 c^2} \frac{\hbar\omega^3 d\omega}{(e^{\hbar\omega/kT} - 1)} \; .$$

Show that the maximum in the distribution is determined by

$$(3 - x)e^x = 3 \quad \text{where} \quad x = \frac{\hbar\omega}{kT} \; .$$ **[6]**

(e) Starting with the expression immediately above, how would you estimate the value of x at which the the black body distribution is a maximum . **[3]**

3. (a) Write down the definition of the definite integral in terms of a limiting procedure of elementary areas. **[3]**

(b) Write down the definition of the indefinite integral. **[3]**

(c) Show that that the derivative of an indefinite integral of $f(x)$ is $f(x)$. **[3]**

(d) From the standard result

$$\frac{d\ x^n}{dx} = nx^{(n-1)} \quad \text{use (c) above to obtain} \quad \int x^n dx = \frac{1}{n+1}x^{n+1} \ .$$ **[3]**

(e) Obtain

$$\int \frac{dx}{x^2} \quad \text{and} \quad \int \frac{dx}{(ax+b)^2} \ .$$ **[3]**

(f) If

$$\frac{d\sigma}{dE} = C\left\{\left(\frac{\Gamma}{2}\right)^2 \frac{1}{\left(\frac{\Gamma}{2}\right)^2 + (E-E_0)^2}\right\} \quad \text{evaluate} \quad \sigma_T = \int_{-\infty}^{\infty} \frac{d\sigma}{dE} dE$$

by making the substitution $(E - E_0) = \frac{\Gamma}{2}\tan\theta$. **[5]**

4. (a) Obtain the Taylor expansion in two variables, as far as the terms shown,

$$f(x,y) = f(a,b) + f_x\Delta x + f_y\Delta y + \frac{1}{2}\left\{f_{xx}\Delta x^2 + 2f_{xy}\Delta x\Delta y + f_{yy}\Delta y^2\right\} \ldots.$$

where $\Delta x = x - a$ and $\Delta y = y - b$. **[4]**

(b) Show that the term involving the second derivatives can be written as

$$\frac{1}{2}\left\{f_{xx}\left(\Delta x + \frac{f_{xy}}{f_{xx}}\Delta y\right)^2 + \Delta y^2\left(f_{yy} - \frac{f_{xy}^2}{f_{xx}}\right)\right\} \ .$$ **[4]**

(c) From this expression explain why the conditions for a local maximum are $f_{xx}, f_{yy} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$. **[4]**

(d) Show that the coordinates of the stationary points of the function

$$f(x,y) = x^3 - 3x^2 - 4y^2$$

are $(0,0)$ and $(2,0)$. **[4]**

(e) Determine the nature of the stationary point $(2,0)$. **[4]**

5. (a) Starting from the definition of the differential df of a function of two variables $f(x,y)$ obtain the expression

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \ .$$ **[6]**

(b) Stationary points are to be found in a function $g(x,y,z)$ subject to the constrain $\phi(x,y,z) = 0$. Explain carefully the method of Lagrange multipliers. **[6]**

(c) The volume of a closed metal can, in the form of a right cylinder of radius R and length L, is to be maximised subject to the total surface of the can being fixed at S . Determine what R must be in terms of L . **[8]**

6. (a) Define the scalar product of two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ and from the definition obtain an expression for the scalar product in cartesian coordinates. **[4]**

(b) An extremely long polymer chain is made from n links, n is very large and each link is of equal length $|\overrightarrow{L_i}|$. Each link $\overrightarrow{L_i}$ is randomly oriented with respect to its predecessor in the chain.

Obtain an expression for $\overrightarrow{R} \cdot \overrightarrow{R}$ where $\overrightarrow{R}$ is the vector from the start to the end of the chain. **[4]**

Determine the mean value of R for the polymer chain. **[1]**

(c) Define the vector product of two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ and from the definition obtain an expression for the vector product in cartesian coordinates. **[5]**

(d) Two straight lines are given by

$$\overrightarrow{r_1} = \lambda(\hat{\imath} + \hat{\jmath} + \hat{k}) \quad \text{and} \quad \overrightarrow{r_2} = \hat{k} + \mu(\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) \ .$$

Determine the shortest distance between these lines. **[6]**

7. (a) From the Maclaurin expansion

$$f(x) = f(0) + \frac{df(0)}{dx}x + \frac{1}{2!}\frac{d^2f(0)}{dx^2}x^2 + \frac{1}{3!}\frac{d^3f(0)}{dx^3}x^3 + \ldots + \frac{1}{n!}\frac{d^nf(0)}{dx^n}x^n$$

obtain the series for

e^x, $\cos\ x$ and, by whatever method, the series for $\sin\ x$. **[6]**

(b)Use these series to derive the Euler relation

$e^{i\theta} = \cos\ \theta + i\sin\ \theta$ and hence obtain de Moivre's theorem $e^{in\theta} = \cos\ n\theta + i\sin\ n\theta$. **[4]**

(c) Show that the sum of the series S_n

$$S_n = e^{i\alpha} + e^{i(\alpha+\theta)} + e^{i(\alpha+2\theta)}, \ldots\ldots\ldots e^{i(\alpha+(n-1)\theta)} = e^{i\alpha}\frac{(1-e^{in\theta})}{(1-e^{i\theta})} = e^{i(\alpha+(n-1)\frac{\theta}{2})}\left(\frac{\sin(\frac{n\theta}{2})}{\sin(\frac{\theta}{2})}\right).$$ **[6]**

(d) Obtain the sum of the series applicable for light from a diffraction grating namely

$$S_n = \cos(\alpha) + \cos(\alpha+\theta) + \cos(\alpha+2\theta), \ldots\ldots\ldots \cos(\alpha+(n-1)\theta).$$ **[4]**

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