Answer FIVE questions.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of marks per sub-section of a question.

1. (a) If the general quadratic equation is of the form $ax^2 + bx + c = 0$ show by 'completing the square' that [4]

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

(b) If the roots are α and β show by any method that [4]

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a} .$$

(c) A particle of energy E and mass m decays into two massless particles of energies E_1 and E_2 respectively. The angle between the trajectories of the two decay particles is ϕ. In such a decay, with c=1,

$$E = E_1 + E_2 \quad \text{and} \quad \sin\frac{\phi}{2} = \frac{m}{2\sqrt{E_1 E_2}} .$$

Show that E_1 satisfies the quadratic equation [4]

$$E_1^2 - EE_1 + \frac{m^2}{4\ \sin^2\frac{\phi}{2}} = 0 .$$

(d) Using any method show that [4]

$$E_1 = \frac{E}{2} \pm \frac{1}{2}\sqrt{E^2 - \frac{m^2}{\sin^2\frac{\phi}{2}}} .$$

(e) At what value of $\sin\frac{\phi}{2}$ is $E_1 = \frac{E}{2}$? [4]

2. (a) Write down the definition of the derivative $\frac{df(x)}{dx}$ in terms of a limiting procedure. **[4]**

(b) By long division show that **[4]**

$$x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + ... + a^{n-2}x + a^{n-1}) \quad (n \geq 2) \ .$$

(c) From the above obtain the standard result that **[4]**

$$\frac{d\ x^n}{dx} = nx^{(n-1)} \ .$$

(d) If **[4]**

$$\overline{E} = \frac{N\epsilon}{1+e^x} \quad \text{where} \quad x = \frac{\epsilon}{kT} \quad \text{and} \quad C_v = \frac{d\overline{E}}{dT} \quad \text{show that} \quad C_v = Nk\frac{x^2 e^x}{(1+e^x)^2} \ .$$

(e) Obtain an expression that determines the stationary values of C_v. **[4]**

3. (a) Write down the definition of the definite integral in terms of a limiting procedure of elementary areas. **[3]**

(b) Write down the definition of the indefinite integral. **[3]**

(c) Show that that the derivative of an indefinite integral of $f(x)$ is $f(x)$. **[3]**

(d) Explain the method of 'integration by parts' and evaluate $\int \ln\ x\ dx$. **[5]**

(e) Given that

$$\int_0^\infty e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \quad \text{evaluate} \quad \int_0^\infty x^2 e^{-\alpha x^2} dx \ .$$

[6]

4. (a) Obtain the Taylor expansion in two variables, as far as the terms shown, **[6]**

$$f(x,y) = f(a,b) + f_x\Delta x + f_y \Delta y + \frac{1}{2}\left\{f_{xx}\Delta x^2 + 2f_{xy}\Delta x \Delta y + f_{yy}\Delta y^2\right\} ....$$

(b) Show that the term involving the second derivatives can be written as **[4]**

$$\frac{1}{2}\left\{f_{xx}\left(\Delta x + \frac{f_{xy}}{f_{xx}}\Delta y\right)^2 + \Delta y^2\left(f_{yy} - \frac{f_{xy}^2}{f_{xx}}\right)\right\} \ .$$

(c) From this expression obtain the conditions for a local maximum. **[4]**

Determine the coordinates of the stationary points of the function **[6]**

$$f(x,y) = x^4 + 4x^2y^2 - 2x^2 + 2y^2 - 1 \ .$$

5. (a) Show that the unit vectors in polar coordinates can be expressed as **[5]**

$$\hat{r} = \hat{\imath}\cos\theta + \hat{\jmath}\sin\theta \quad \text{and} \quad \hat{\theta} = -\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta \ .$$

(b) Starting from the position vector in polar coordinates $\overrightarrow{r} = r\hat{r}$ show that the velocity is given by **[5]**

$$\frac{d\vec{r}}{dt} = \frac{dr}{dt}\hat{r} + r\frac{d\theta}{dt}\hat{\theta} = v_r\hat{r} + v_\theta\hat{\theta} \ .$$

(c) A boat is to dock with a ship. The ship is sailing along a straight course with speed v. The boat moves with constant speed nv, its motion being always directed towards the ship. Show that the polar equation of the course of the boat as observed from the ship is given by

$$\frac{A}{r} = \sin\theta \tan^n \frac{\theta}{2}$$

where A is a constant, the origin of coordinates is the ship and the x axis is in the direction of the ship's motion. **[10]**

You may use without proof that $\int \text{cosec}\ \theta d\theta = \ln\ \tan\frac{\theta}{2}$.

6. (a) Define the scalar product of two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ and from the definition obtain an expression for the scalar product in cartesian coordinates. **[4]**

(b) Define the vector product of two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ and from the definition obtain an expression for the vector product in cartesian coordinates. **[6]**

(c) If $\overrightarrow{p}$ is the position vector of a point, $\overrightarrow{a}$ is the position vector of a point in a plane and $\hat{n}$ is the unit vector normal to the plane, show that the perpendicular distance d from the point to the plane is given by **[4]**

$$d = (\overrightarrow{p} - \overrightarrow{a}) \cdot \hat{n}$$

(d)Find the distance from the point P $(1, 2, 3)$ to the plane which contains the points $A(0, 1, 0)$, $B(2, 3, 1)$ and $C(5, 7, 2)$. **[6]**

7. (a) From the Maclaurin expansion

$$f(x) = f(0) + \frac{df(0)}{dx}x + \frac{1}{2!}\frac{d^2f(0)}{dx^2}x^2 + \frac{1}{3!}\frac{d^3f(0)}{dx^3}x^3 + \ldots + \frac{1}{n!}\frac{d^nf(0)}{dx^n}x^n$$

obtain the series for **[6]**

$e^x, \cos\ x$ and, by whatever method, the series for $\sin\ x$.

(b)Use these series to derive the Euler relation **[6]**

$e^{i\theta} = \cos\ \theta + i\sin\ \theta$ and de Moivre's theorem $e^{in\theta} = \cos\ n\theta + i\sin\ n\theta$.

(c) Find an expression for $\cos^3\theta$ in terms of the cosines of multiple angles. **[4]**

(d) Evaluate $(-8)^{\frac{1}{3}}$. **[4]**

END OF PAPER