Stellar Structure and Astrophysical Processes:

Notes for PHAS3135 [and my legacy notes from PHAS2112]

Preface

These notes are intended for use with Part II of PHAS3135, The Physics of Stars. A pre-requisite of this course is PHAS2112, which I used to teach, and for which I developed substantial notes, including additional material for my own interest. The 3135 syllabus includes some of that additional material, so what follows is a distinct, self-contained set of 3135 notes (partly drawn from my extended 2112 notes), followed by my original 2112 notes (starting here as Chapter 11), provided both as an 'aide-memoire' for the student, and to allow me to explicitly reference 2112 topics where necessary.

Do bear in mind that these are *notes,* written principally for my own use. Boxed items, and indented or small-font sections, are 'extras', not part of the syllabus (and therefore not examinable). You will find sections of incomplete text flagged up; some repetition (especially between the '3135' and '2112' notes); a numbering system that probably departs from lectures (as the written notes are under continual revision); and, probably, outright errors. If you think you've found any mistakes, please let me know.

Contents

PHAS3135 – Part II, Stellar Structure & Evolution

Section 1

The Equations of Stellar Structure

1.1 Introduction: motivation

In this part of the course, we consider aspects of the internal operation of (principally) single stars: their structure and evolution. Our overarching aim in this is to interpret observations such as the Hertzsprung–Russell diagrams shown in Fig. 1.1

For the present purposes, we use a working definition of a star as an isolated body that is bound by self-gravity, and which radiates energy supplied by an internal source. Self-gravity ensures that the star is approximately spherical (rotation introduces centrifugal forces which, for sufficiently fast rotation, may introduce distortions); the internal source of energy is nuclear fusion for most of the stellar lifetime (although for, e.g., white dwarfs, stored thermal energy is responsible for the observed luminosity).

The essence of stellar structure is the competition between the force of gravity, which always wants to make a star collapse, and the outward force of pressure. For almost the entire lifetime of a star, these forces are in balance; the star is in (or very close to) hydrostatic equilibrium, but as internal energy is released, the internal composition, and hence structure, must evolve. Thus 'stellar structure' and 'evolution' are intimately linked.

1.2 Review: Basic Equations of Stellar Structure

For reference, we remind ourselves of the basic equations of stellar structure, introduced in PHAS 2112 and Dr. Zane's notes:

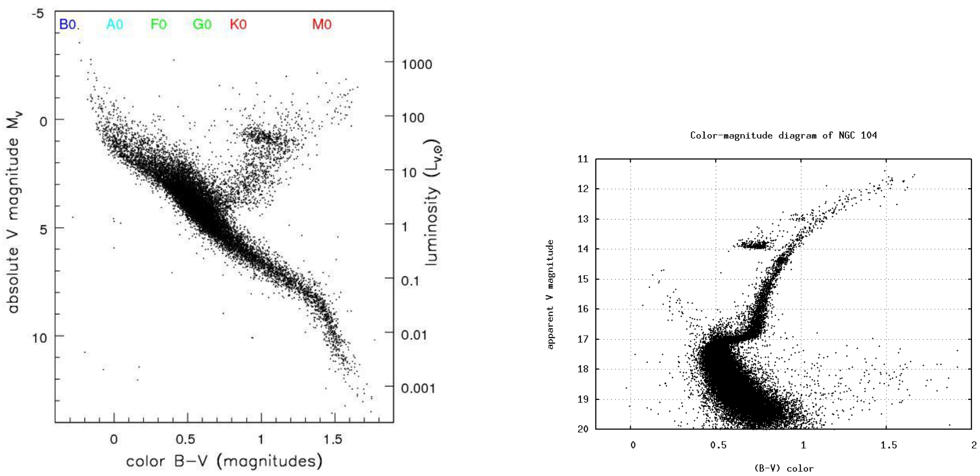


Figure 1.1: Hertzsprung–Russell (colour–magnitude) diagrams. Left, *Hipparcos* volume-limited sample (stars of different ages); right, *HST* observations of the globular cluster 47 Tuc (coeval sample).

1.2.1 Hydrostatic (pressure) equilibrium

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = \frac{-Gm(r)\rho(r)}{r^2} = -\rho(r)\,g(r) \tag{1.1}$$

or

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} + \rho(r)\,g(r) = 0$$

[In the supplementary 2112 notes, eqtn. 20.1; this numbering may well differ from that in use when you took in PHAS 2112].

The principal sources of pressure throughout a 'normal' (non-degenerate) star are gas pressure, and radiation pressure.[1] We will take the corresponding equations of state to be, in general,

$$\begin{aligned} P_{\mathrm{G}} &= nkT; \\ &= (\rho kT)/(\mu m(\mathrm{H})) \qquad (1.2) \\ P_{\mathrm{R}} &= \frac{1}{3}aT^4 \qquad (1.3) \end{aligned}$$

for number density n at temperature T, density ρ; μ is the mean molecular weight, and $m(\mathrm{H})$ the hydrogen mass; a is the radiation constant, $a = 4\sigma/c$; with σ the Stefan-Boltzmann constant, and k Boltzmann's constant.

[1] Electron degeneracy pressure is important in white dwarfs, and neutron degeneracy pressure in neutron stars.

1.2.2 Mass Continuity

For spherical symmetry, the equation of mass continuity for static configurations is

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho(r) \tag{1.4}$$

[PHAS 2112 eqtn. 20.3; in a spherically symmetric flow, such as may apply in a stellar wind with mass-loss rate $\dot{M}$, we instead have

$$\dot{M} = 4\pi r^2 \rho(r) v(r)$$

where $\rho(r), v(r)$ are the density and (radial) flow velocity at radius r.]

1.2.3 Energy continuity

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2\, \rho(r)\epsilon(r) \tag{1.5}$$

[PHAS 2112 eqtn. 20.7].

where

- r is radial distance measured from the centre of the star
- $P(r)$ is the total pressure at radius r
- $\rho(r)$ is the density at radius r
- $g(r)$ is the gravitational acceleration at radius r
- $m(r)$ is the mass contained with radius r
- $L(r)$ is the total energy transported through a spherical surface at radius r
- $\epsilon(r)$ is the energy generation rate per unit mass at radius r

The stellar radius is R, the stellar mass is $M \equiv m(R)$, and the emergent luminosity $L \equiv L(R)$ (dominated by radiation at the stellar surface).

Section 2

Energy transport

We suppose stars to be hotter inside than outside (so that, e.g., nuclear fusion may occur), so there must be an energy flow. We are familiar with three basic mechanisms of energy transport:
– radiation
– convection
– conduction

In the context of stellar astrophysics, conduction is important only under the degenerate conditions found in white dwarfs and neutron stars (since gases in general are poor conductors). For 'normal' stars, therefore, the key processes transporting energy are radiation and convection.

Radiative transport: Energy is transported by photons. In stellar interiors the opacities are high, and the mean free path correspondingly low – about 1 mm in the case of the Sun (Section 22.4). In this sense, the radiation doesn't *flow* outwards, but rather *diffuses* outwards.

Convective transport: If the radiation is unable to escape a layer at a rate that matches the energy input, then 'something's got to give'. What gives is the static nature of the layer: convection is initiated and starts to transport energy. This suggests that hydro*static* equilibrium breaks down, but the dynamical timescale is short compared to the flow timescale (Section 22), so in practice HSE continues to be an excellent approximation.

The nett energy flux is, under most circumstances, simply the sum of radiative and convective terms,

$$L(r) = L_{\rm rad}(r) + L_{\rm cnv}(r)$$

2.1 Radiative energy transport in stellar interiors

In order to determine under what circumstances convection is important, we first evaluate how much energy can be transported by radiation alone.

2.1.1 The equation of radiative transfer in stellar interiors

In optically thick environments – in particular, stellar interiors – radiation is often the most important transport mechanism, but, to repeat, for large opacities the radiant energy doesn't flow directly outwards; instead, it *diffuses* slowly outwards.

To express this transport quantitatively, the same general principles may be applied as led to the equation of radiative transfer in plane-parallel stellar atmospheres

$$\mu \frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = I_\nu - S_\nu. \tag{2.1}$$

[PHAS 2112 eqtn. (14.6), and Dr. Zane's lectures]. In the interior the photon mean free path is (very) short compared to the radius, so 'plane parallel' is fine.

We recall that, in general, the intensity, I_ν, at some position is direction-dependent; i.e., is $I_\nu(\theta, \phi)$ (although the explicit angular dependence is generally dropped for economy of nomenclature); the same is true in principle of the source function, although in practice any such dependence is negligible. Multiplying eqtn. (2.1) by $\mu \equiv \cos\theta$ and integrating over solid angle, using $\mathrm{d}\Omega = \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi = \mathrm{d}\mu \,\mathrm{d}\phi$, then

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\nu} \int_0^{2\pi} \int_{-1}^{+1} \mu^2 I_\nu(\mu, \phi) \,\mathrm{d}\mu \,\mathrm{d}\phi = \int_0^{2\pi} \int_{-1}^{+1} \mu I_\nu(\mu, \phi) \,\mathrm{d}\mu \,\mathrm{d}\phi - \int_0^{2\pi} \int_{-1}^{+1} \mu S_\nu(\mu, \phi) \,\mathrm{d}\mu \,\mathrm{d}\phi;$$

The radiation field in the interior is axially symmetric, (i.e., no azimuthal dependence), so $\int_0^{2\pi} \mathrm{d}\phi = 2\pi$ on both sides, and cancels, whence

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\nu} \int_{-1}^{+1} \mu^2 I_\nu(\mu) \,\mathrm{d}\mu = \int_{-1}^{+1} \mu I_\nu(\mu) \,\mathrm{d}\mu - \int_{-1}^{+1} \mu S_\nu(\mu) \,\mathrm{d}\mu. \tag{2.2}$$

The first two terms should be familiar in the context of moments of the radiation field:

$$\frac{1}{2} \int_{-1}^{+1} \mu^2 I_\nu(\mu) \,\mathrm{d}\mu = K_\nu \tag{2.3}$$

[the second-order moment, or K integral; PHAS 2112 eqtn. (11.14)] and

$$2\pi \int_{-1}^{+1} \mu I_\nu(\mu) \,\mathrm{d}\mu = F_\nu \tag{2.4}$$

$$= 4\pi H_\nu$$

[where H_ν is the Eddington flux, or first-order moment of the radiation field; PHAS 2112 eqtn. (11.9)]. This is progress: the flux is the rate of energy transport, which is what we're seeking to evaluate.

Since the radiation field is locally isotropic to a very good approximation (see Box 2.2) we can take I_ν out of the K integral, whence

$$\begin{aligned} K_\nu &= \frac{1}{2}\frac{\mu^3}{3} I_\nu \Big|_{-1}^{+1} \\ &= \frac{1}{3} I_\nu \qquad\qquad \left[\equiv \frac{1}{3} J_\nu \text{ for isotropy}\right] \end{aligned} \tag{2.5}$$

[PHAS 2112 eqtn. (11.15)].

Using these equations for the first two terms in eqtn. (2.2), and assuming that the emissivity also has no preferred direction (true to an excellent approximation in stellar interiors) so that the source function is isotropic (i.e., so that the final term in eqtn. 2.2 is zero), we obtain

$$\frac{\mathrm{d}K_\nu}{\mathrm{d}\tau_\nu} = \frac{F_\nu}{4\pi}$$

or, from eqtn. (2.5),

$$\frac{1}{3}\frac{\mathrm{d}I_\nu}{\mathrm{d}\tau_\nu} = \frac{F_\nu}{4\pi}.$$

Because the photon mean free paths are very short, conditions in the interior are very close to local thermodynamic equilibrium (LTE; PHAS 2112 Sec. 21.1). In LTE we may set $I_\nu = B_\nu(T)$, the Planck function; and by definition, $\mathrm{d}\tau_\nu = -k_\nu\,\mathrm{d}r$ (where the minus arises because the optical depth is measured inwards, and decreases with increasing r). Making these substitutions, and integrating over frequency,

$$\int_0^\infty F_\nu\,\mathrm{d}\nu = -\frac{4\pi}{3}\int_0^\infty \frac{1}{k_\nu}\frac{\mathrm{d}B_\nu(T)}{\mathrm{d}T}\frac{\mathrm{d}T}{\mathrm{d}r}\mathrm{d}\nu \tag{2.6}$$

To simplify this further, we introduce the *Rosseland mean opacity,* $\overline{k}_\mathrm{R}$ $(= \overline{\kappa}_\mathrm{R}\rho)$,[1,2] defined by

$$\frac{1}{\overline{k}_\mathrm{R}}\int_0^\infty \frac{\mathrm{d}B_\nu(T)}{\mathrm{d}T}\mathrm{d}\nu = \int_0^\infty \frac{1}{k_\nu}\frac{\mathrm{d}B_\nu(T)}{\mathrm{d}T}\mathrm{d}\nu;$$

conveniently, $\overline{k}_\mathrm{R}$ can be evaluated, as a function of temperature and density, separately from any other factors.

[1] Recall that opacity may be expressed in several ways, most commonly as 'per unit mass' or 'per unit volume'. We use k to denote opacity per unit volume, and κ where reference is made to opacity per unit mass; clearly, then, $k = \kappa\rho$.

[2] The Rosseland mean opacity represents the harmonic mean of k_ν, weighted by $\mathrm{d}B_\nu(T)/\mathrm{d}T$. This weighting factor is small for very low and very high frequencies, and peaks at $\nu_\mathrm{p} = 4kT/h$.

Recalling that

$$\int_0^\infty \pi B_\nu \, \mathrm{d}\nu = \sigma T^4 \qquad \text{(PHAS 2112, eqtn. (11.21))}$$

we also have

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}B_\nu(T)}{\mathrm{d}T}\mathrm{d}\nu &= \frac{\mathrm{d}}{\mathrm{d}T}\int_0^\infty B_\nu(T)\mathrm{d}\nu \\ &= \frac{4\sigma T^3}{\pi}\end{aligned}$$

(at given T) so that eqtn. (2.6) can be written as

$$\int_0^\infty F_\nu \, \mathrm{d}\nu = -\frac{4\pi}{3}\frac{1}{\bar{k}_\mathrm{R}}\frac{\mathrm{d}T}{\mathrm{d}r}\frac{acT^3}{\pi} \tag{2.7}$$

where a is the radiation constant, $4\sigma/c$; that is, the total (frequency-integrated) radiant energy flux is

$$F = -\frac{4\pi}{3}\frac{1}{\bar{k}_\mathrm{R}}\frac{\mathrm{d}T}{\mathrm{d}r}\frac{acT^3}{\pi}, \tag{2.8}$$

(with no direct dependence on the rate of energy generation!). The minus sign simply means that the energy flows in the opposite direction to the temperature gradient (i.e., outwards, not inwards, for stars).

[Incidentally, eqtn. 2.8 shows that radiative diffusion is completely analogous to conduction;

$$F \propto \frac{\mathrm{d}T}{\mathrm{d}r},$$

which is equivalent to Fourier's law of thermal conduction.]

Equation 2.8 is our adopted form for the radiative flux, or transport of energy by radiation. It may be applied in environments where the photon mean free path is short compared to the scales over which physical parameters (notably temperature) change; it therefore becomes inappropriate as the stellar surface is approached, where a more detailed approach to radiative transfer is required.

Box 2.1. The radiative energy density is $U = aT^4$ [PHAS 2112, eqtn. (11.27)], so that $\mathrm{d}U/\mathrm{d}T = 4aT^3$, and we can express eqtn. (2.7) as

$$\begin{aligned} F &= \int_0^\infty F_\nu \,\mathrm{d}\nu \\ &= -\frac{c}{3\bar{k}_\mathrm{R}} \frac{\mathrm{d}T}{\mathrm{d}r} \frac{\mathrm{d}U}{\mathrm{d}T} \\ &= -\frac{c}{3\bar{k}_\mathrm{R}} \frac{\mathrm{d}U}{\mathrm{d}r} \end{aligned}$$

This 'diffusion approximation' shows explicitly how the radiative flux relates to the energy-density gradient; the constant of proportionality, $c/3\bar{k}_\mathrm{R}$, is called the diffusion coefficient. The larger the opacity, the less the flux of radiative energy, as one might intuitively expect.

2.1.2 Radiative temperature gradient

The stellar luminosity at some radius r is given by

$$L(r) = 4\pi r^2 \int_0^\infty F_\nu \,\mathrm{d}\nu$$

so, from eqtn. 2.8

$$L(r) = -\frac{16\pi}{3} \frac{r^2}{\bar{k}_\mathrm{R}} \frac{\mathrm{d}T}{\mathrm{d}r} acT^3, \tag{2.9}$$

We can simply rearrange eqtn. (2.9) to express the temperature gradient where energy transport is radiative:

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{16\pi} \frac{\bar{k}_\mathrm{R}}{r^2} \frac{L(r)}{acT^3} = -\frac{3}{16\pi} \frac{\bar{\kappa}_\mathrm{R}\rho(r)}{r^2} \frac{L(r)}{acT^3}. \tag{2.10}$$

We'll need this in a slightly different form later on, so combining this result with hydrostatic equilibrium,

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = \frac{-Gm(r)\rho(r)}{r^2}, \tag{1.1}$$

we obtain

$$\frac{\mathrm{d}T}{\mathrm{d}P} = -\frac{3\bar{\kappa}_\mathrm{R} L(r)}{16\pi \, acT^3 \, Gm(r)} \tag{2.11}$$

or equivalently,

$$\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} = -\frac{3\bar{\kappa}_\mathrm{R} L(r) P}{16\pi \, acT^4 \, Gm(r)} \tag{2.12}$$

(a form that we'll use in Section 2.3).

2.1.3 Von Zeipel's law (not in lectures)

From eqtn. (2.7),

$$F \propto \frac{T^3}{\overline{\kappa}_{\rm R}\,\rho}\,\frac{{\rm d}T}{{\rm d}r} \qquad (2.13)$$

$$\propto \frac{T^3}{\overline{\kappa}_{\rm R}\,\rho}\,\frac{{\rm d}T}{{\rm d}\psi}\,\frac{{\rm d}\psi}{{\rm d}r} \qquad (2.14)$$

where ψ is the gravitational potential (and hence ${\rm d}\psi/{\rm d}r$ is the local gravity,[3] g). In hydrostatic equilibrium (see eqtn. 1.1)

$$\frac{{\rm d}P}{{\rm d}r} = -\rho(r)g(r) \propto \rho\frac{{\rm d}\psi}{{\rm d}r} \qquad (1.1)$$

so that the pressure P is a function of the potential ψ – and hence the density must also be a function of ψ.[4] For an equation of state of the general form

$$T = T(P, \rho) \qquad (2.15)$$

we therefore see that T must also be a function of ψ,

$$T = T(\psi). \qquad (2.16)$$

The coeffiecent of ${\rm d}\psi/{\rm d}r$ in eqtn. (2.14) is therefore a function of ψ alone, whence

$$F \propto \frac{{\rm d}\psi}{{\rm d}r} \propto g \qquad (2.17)$$

or, equivalently,

$$T_{\rm eff} \propto g^{0.25} \qquad (2.18)$$

which is known as von Zeipel's law. Although it relies on the assumption of radiative energy transport by diffusion, which breaks down in a stellar atmosphere, the atmosphere is usually very thin compared to the radiative envelope, so even the *surface* flux can be expected to obey eqtn. (2.18) for stars in hydrostatic equilibrium and for which energy transport through the outer envelope is radiative.

Von Zeipel's law is of particular interest for close binary stars and rapidly rotating single stars. In either case, the local gravity, and hence the local temperature, can vary over the stellar surface (which is conventionally defined by a constant potential). Although increasing gravity results in increasing flux, the practical effects have come to be known as gravity *darkening*, because rapid rotation, or a close companion star, both serve to reduce a star's local gravity (and hence reduce the temperature locally).

It's of interest that von Zeipel also demonstrated that a rotating star *cannot* be simultaneously in strict hydrostatic and radiative equilibrium, undermining the basis of his

[3]In circumstances where von Zeipel's law is important, gravity is, in general, not a central force, so we should actually set $g = \nabla\psi$; but the central-force approximation is adequate for our purposes (and the correct general result is obtained).

[4]Since ρ is a scalar, the gradients of P and ψ are everywhere parallel.

'law'. In practice, as shown by Eddington and by Sweet, rotation induces circulation currents in the stellar interior; however, these currents are sufficiently slow as to not lead to *significant* departures from hydrostatic equilibrium (the circulation timescales are long compared to the dynamical timescales discussed in Section 22.1.1), and gravity darkening is observed to occur in practice.

2.2 Mass–Luminosity Relationship

We can now put together our basic stellar-structure relationships to demonstrate a scaling between stellar mass and luminosity. From hydrostatic equilibrium,

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = \frac{-Gm(r)\rho(r)}{r^2} \quad \rightarrow \quad P \propto \frac{M}{R}\rho \tag{1.1}$$

but our (gas) equation of state is $P = (\rho kT)/(\mu m(\mathrm{H}))$, so

$$T \propto \frac{\mu M}{R}$$

(and $\mathrm{d}T/dcR \propto R^{-2} \propto T/R$). For stars in which the dominant energy transport is radiative, we have

$$L(r) \propto \frac{r^2}{\bar{\kappa}_\mathrm{R}}\frac{\mathrm{d}T}{\mathrm{d}r}T^3 \qquad \propto \frac{r^2}{\overline{\kappa}_\mathrm{R}\rho(r)}\frac{\mathrm{d}T}{\mathrm{d}r}T^3 \tag{2.9}$$

so at the surface ($r = R$)

$$L \propto \frac{RT^4}{\overline{\kappa}_\mathrm{R}\rho}.$$

From mass continuity (or by inspection) $\rho \propto M/R^3$, giving

$$\begin{aligned} L &\propto \frac{R^4T^4}{\overline{\kappa}_\mathrm{R}M} \\ &\propto \frac{R^4}{\overline{\kappa}_\mathrm{R}M}\left(\frac{\mu M}{R}\right)^4; \end{aligned}$$

i.e.,

$$L \propto \frac{\mu^4}{\overline{\kappa}_\mathrm{R}}M^3.$$

This simple dimensional analysis (which makes *no assumptions* about energy sources) yields a dependency which is in quite good agreement with observations; for solar-type main-sequence stars, the empirical mass–luminosity relationship is $L \propto M^{3.5}$.

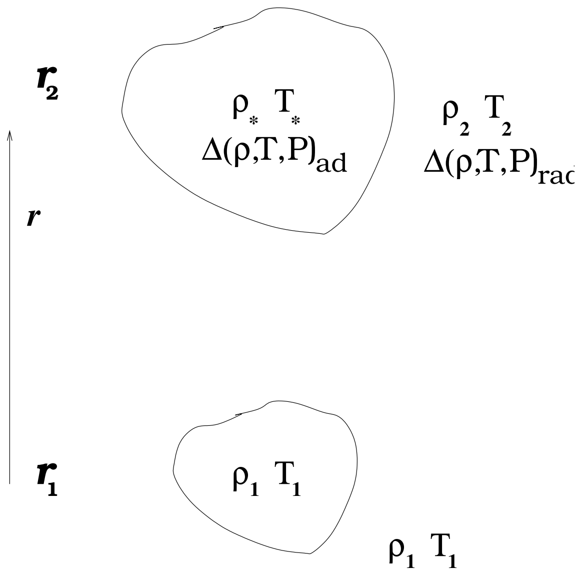


Figure 2.1: A (potentially) convective 'blob' in a stellar envelope, defining the terminology used in Section 2.3.

2.3 Convection in stars

2.3.1 Schwarzschild criterion

For convection to occur, there must be some temperature gradient (in the case of stars, a radial temperature gradient). We have seen that the temperature gradient is given by

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{3}{16\pi}\frac{\overline{k}_{\mathrm{R}}}{r^2}\frac{L(r)}{acT^3} \tag{2.10}$$

where energy transport is radiative; that is, high opacity leads to large temperature gradients (as we might expect intuitively; the opacity blocks the flow of radiant energy from hotter to cooler regions). If the energy flux isn't contained by the temperature gradient, we have to invoke another mechanism – convection – for energy transport (recall, conduction is negligible in ordinary stars.) Under what circumstances will this arise? Karl Schwarzschild[5] (1906) developed a standard criterion for determining if convection occurs or not. (Here we'll derive it as a criterion for stability, although we could equally well establish a criterion for instability.)

To follow Schwarzschild's reasoning, we suppose that we start with a stellar envelope in radiative equilibrium – in some sense, its 'natural state' – and that, through some minor perturbation, an element (or cell, or blob, or bubble) of gas is displaced upwards within a star.

[5] Perhaps better know for finding the first exact solutions to the field equations of Einstein's general theory of relativity, leadin to the 'Schwarzschild radius' for the event horizon of nonrotating black holes.

Our essential assumptions are that the cell rises slowly enough that it remains in pressure equilibrium (essentially, that it moves subsonically, so that hydrostatic equilibrium is maintained); but fast enough that it doesn't lose energy to its surroundings (i.e., that the cell cools adiabatically, while that the ambient temperature is determined by radiative equilibrium).

As the cell rises into a lower-pressure regime, it will expand to bring it into pressure equilibrium with the surroundings (a process whose timescale is naturally set by the speed of sound and the linear scale of the perturbation), but not, in general, into thermal equilibrium; that is, its pressure, but not its density and temperature, will match conditions in the surrounding gas. If it cell gas is less dense, then simple buoyancy comes into play; the cell will continue to rise, and convective motion occurs.[6]

The essence of the Schwarzschild criterion is thus: we obtain *stability* (rising cell denser than surroundings[7]) if

$$|\Delta\rho_{\rm ad}| < |\Delta\rho_{\rm rad}|$$

(where the 'ad', 'rad' subscripts indicate adiabatic and radiative conditions). Since

$$\Delta\rho = \left(\frac{{\rm d}\rho}{{\rm d}r}\right)\Delta r$$

(and Δr is the same for the cell and the ambient gas) we can express this statement as

$$\left|\frac{{\rm d}\rho}{{\rm d}r}\right|_{\rm ad} < \left|\frac{{\rm d}\rho}{{\rm d}r}\right|_{\rm rad} \tag{2.19}$$

The Schwarzschild criterion is conventionaly expressed in terms of temperature gradients (rather than density gradients). We therefore use our assumption of pressure equilibrium; since the pressure is the same inside the cell and in the ambient gas at both between r_1 and r_2, so

$$\Delta P_{\rm ad} = \Delta P_{\rm rad};$$

but $P \propto \rho T$ (equation of state, eqtn. 1.2), so

$$\Delta\rho_{\rm ad}T_{\rm ad} = \Delta\rho_{\rm rad}T_{\rm rad}.$$

In other words, an increase in density is matched by a decrease in temperature, hence

$$\left|\frac{{\rm d}T}{{\rm d}r}\right|_{\rm ad} > \left|\frac{{\rm d}T}{{\rm d}r}\right|_{\rm rad} \tag{2.20}$$

[6] Another way of looking at this is that the entropy (per unit mass) of the blob is conserved, so the star is unstable if the ambient entropy per unit mass decreases outwards.

[7] We could follow identical arguments for stability by requiring a descending cell to be less dense than its surroundings

is equivalent to eqtn. (2.19) – i.e., is the condition for stability.

Finally, we invoke the equation of hydrostatic equlibrium

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -\rho(r)\, g(r) \tag{1.1}$$

and the (gas-pressure) equation of state,

$$P = (\rho k T)/(\mu m(\mathrm{H})) \tag{1.2}$$

to write

$$\begin{aligned}
\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right| &\equiv \left|\frac{\mathrm{d}T}{\mathrm{d}P}\frac{\mathrm{d}P}{\mathrm{d}r}\right| \\
&= \left|\frac{\mathrm{d}T}{\mathrm{d}P}\, g\rho\right| && (HSE) \\
&= \left|\frac{\mathrm{d}T}{\mathrm{d}P}\right|\, g\frac{\mu m(\mathrm{H})}{kT} P && (EOS) \\
&= \left|\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right|\, g\frac{\mu m(\mathrm{H})}{k} && (2.21)
\end{aligned}$$

Substituting this into eqtn. (2.20) we obtain

$$\left|\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right|_{\mathrm{ad}} > \left|\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right|_{\mathrm{rad}}.$$

This is frequently written in the more compact notation

$$\nabla_{\mathrm{ad}} > \nabla_{\mathrm{rad}}, \tag{2.22}$$

which tells us that if the temperature gradient in the stellar envelope is larger than the adiabatic temperature gradient, convection occurs.

The astute student will've noticed that we didn't really need to suppose that the ambient conditions are radiative; only that they exist(!), and differ from convective conditions. So we can write, more generally, that a condition for *in*stability is

$$\nabla_{\mathrm{ad}} < \nabla, \tag{2.23}$$

which is the form in which the *Schwarzschild criterion for convective instability* is often expressed. It tells us that if a *superadiabatic temperature gradient* exists (i.e., if the actual temperature gradient exceeds the adiabatic value, given by eqtn. 2.24, below), convection will occur.

2.3.2 When do superadiabatic temperature gradients occur?

Since large temperature gradients arise in (initially) radiative envelopes if the opacity is high (eqtn. 2.10), we interpret this as meaning that convection occurs when the opacity is too high for radiative transport to be efficient. To demonstrate this analytically we appeal to thermodynamic arguments.

Under the adiabatic conditions appropriate to our rising cell,

$$PV^\gamma = \text{constant}$$

where $\gamma = C_P/C_V$, the ratio of specific heats at constant pressure and constant volume. Thus, for a gas cell of constant mass ($V \propto \rho^{-1}$),

$$\begin{aligned} P &\propto \rho^\gamma; && \text{but also} \\ P &\propto \rho T, && \text{whence} \\ P^{\gamma-1} &\propto T^\gamma \end{aligned} \tag{1.2}$$

or

$$\left|\frac{\mathrm{d}(\ln T)}{\mathrm{d}(\ln P)}\right|_{\text{ad}} = \frac{\gamma - 1}{\gamma} \tag{2.24}$$

The Schwarzschild criterion for *stability*

$$\left|\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right|_{\text{rad}} < \left|\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right|_{\text{ad}} \tag{2.22}$$

can therefore be written as

$$\nabla_{\text{rad}} < \frac{\gamma - 1}{\gamma} \tag{2.25}$$

or, in terms of temperature gradient (cp. eqtn. 2.21)

$$\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\text{rad}} < \frac{\gamma - 1}{\gamma}\left|\frac{T}{P}\frac{\mathrm{d}P}{\mathrm{d}r}\right|. \tag{2.26}$$

We know the radiative temperature gradient (eqtn. 2.10); whence, by reference to eqtns (2.12) and (2.24), the Schwarzschild criterion for *stability* can be written as

$$\frac{3\overline{\kappa}_{\text{R}} L(r) P}{16\pi a c T^4 G m(r)} < \frac{\gamma - 1}{\gamma}. \tag{2.27}$$

This shows us explicitly that the product of the luminosity and the opacity must be 'small' (in some sense) if the system is to be stable against convection, as we've already asserted on intuitive physical grounds.

Finally, rearranging and differentiating the equation of state $P = nkT = (\rho kT)/(\mu m(\mathrm{H}))$ gives

$$\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P} = 1 + \frac{\mathrm{d}\ln \mu}{\mathrm{d}\ln P} - \frac{\mathrm{d}\ln \rho}{\mathrm{d}\ln P}, \tag{2.28}$$

which demonstrates that compositional changes can also influence whether or not convection occurs (separately to the effect of composition on the opacities). This can be important wherever there is a 'mu gradient' (i.e., composition change), such as at the boundary between the stellar core and envelope.

2.3.3 Physical conditions associated with convection

From equations 2.25–2.28 we can see several ways in which convection may, in principle be induced, but eqtn. (2.25) argues that the essential requirements are either:
∇_{rad} becomes large (compared to ∇_{ad}), or
∇_{ad} becomes small (compared to ∇_{rad}). Alternatively, from eqtn. (2.27), we can see what this means in terms of luminosity, opacity, and the adiabatic exponent γ.

In nature, convectively unstable regions occur:

(i) In the cores of massive stars, where the radiation flux $L(r)/4\pi r^2$ can be very large, driving convection.

(ii) Where the opacity is too great to allow the radiation to flow at an equilibrium rate (e.g., protostars).

(iii) In the envelopes of cool stars, where the adiabatic exponent γ can approach unity (and hence $(\gamma - 1)/\gamma$ can become very small; the γ effect).

For a monatomic ideal gas (representative of stellar interiors),[8] $\gamma = 5/3$ and so $(\mathrm{d}\ln T/\mathrm{d}\ln P)_{\mathrm{ad}} = 0.4$, but under changing conditions of ionization this exponent changes. For a simple pure-hydrogen composition it can be shown that

$$\left|\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right| = \frac{2 + X(1-X)\left((5/2) + E_1/(kT)\right)}{5 + X(1-X)\left((5/2) + E_1/(kT)\right)^2}$$

where

$$X = \frac{n_{\mathrm{e}}}{n_{\mathrm{P}} + n(\mathrm{H}^0)}$$

is the degree of ionization, and E_1 is the ionization potential. For $X = 0$ or 1, this recovers $(\mathrm{d}\ln T/\mathrm{d}\ln P)_{\mathrm{ad}} = 0.4$, but in regions of partial ionization lower values apply, with a minimum at $X = 0.5$ $[(\mathrm{d}\ln T/\mathrm{d}\ln P)_{\mathrm{ad}} = 0.07]$ which occurs (e.g.) near the base of the solar photosphere.

[8] Radiation obeys a 'gas law' with $\gamma = 4/3$

The switch from radiative core/convective envelope to convective core/radiative envelope occurs on the main sequence at masses only very slightly more than the Sun's. This is related to the core energy-generation mechanism, as the principal hydrogen-burning process switches from proton-proton chains (which generate energy at a rate that can be transported radiatively) to CNO processing.

2.4 Convective energy transport: mixing-length 'theory'

So far, we have only tested whether or not convection is likely to occur; we have not addressed how energy is transported by this mechanism – i.e., we don't yet know the convective flux. Unfortunately, convection is a complex, hydrodynamic process. Although much progress is being made in numerical modelling of convection over short timescales, it's not feasible at present to model convection in detail in stellar-evolution codes routinely, because of the vast disparities between convective and evolutionary timescales. Instead, we appeal to simple parameterizations of convection, of which mixing-length 'theory' is the most venerable, and the most widely applied.

We suppose that
(i) the envelope becomes convectively unstable at some radius r_0, and that a convective cell then rises, hydrostatically and adiabatically, through some characteristic distance ℓ – the *mixing length*;
(ii) the excess thermal energy of the cell is released into the ambient medium; and
(iii) the cooled cell sinks back down (or, if you prefer, is displaced downwards by the next rising cell).
Because we are moving energy from deeper to shallower regions, the temperature gradient is shallower for the cell than the pure radiative case.

From hydrostatic equilibrium (eqtn. 1.1) and the perfect gas equation (eqtn. 1.2) we have

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -gP\frac{\mu m(\mathrm{H})}{kT}, \qquad \text{or}$$
$$\frac{\mathrm{d}P}{P} = -g\frac{\mu m(\mathrm{H})}{kT}\mathrm{d}r, \equiv -\frac{\mathrm{d}r}{H}. \tag{2.29}$$

The solution of eqtn. 2.29 is

$$P = P_0 \exp\left(-r/H\right)$$

so H, the pressure scale height, is the vertical distance over which the pressure drops by a factor e. The mixing length is conveniently expressed in terms of this scale height; typically, we expect $\ell \simeq H$, but, since the detailed physics is not well understood, a scale factor (or fudge factor!) is usually introduced, whereby

$$\ell = \alpha H,$$

with $\alpha \sim 0.5$–1.5.

For simplicity (given the weakness of other assumptions), we suppose that ℓ is the same for all cells, and that the velocity of all cells is also the same.

For a cell moving with velocity v the flux of energy across unit area is given by the mass flux times the heat energy per unit mass:

$$\begin{aligned} F_{\rm conv} &= \rho v \times {\rm d}Q \\ &= \rho v \times C_{\rm P} \Delta T \end{aligned} \tag{2.30}$$

where $C_{\rm P}$ is the specific heat at constant pressure. To progress we need an estimate of the velocity v, which we obtain by considering the buoyancy force,

$$f_{\rm b} = -g \Delta\rho. \tag{2.31}$$

Here $\Delta\rho$ is the density difference between the cell and ambient gas which we can determine from the equation of state, eqtn. 1.2,

$$\frac{\Delta P}{P} = \frac{\Delta\rho}{\rho} + \frac{\Delta T}{T} - \frac{\Delta\mu}{\mu}. \tag{2.32}$$

In hydrostatic pressure equlibrium $\Delta P = 0$, whence

$$\frac{\Delta\rho}{\rho} = \frac{\Delta\mu}{\mu} - \frac{\Delta T}{T}$$

or

$$\Delta\rho = -\rho \frac{\Delta T}{T} \left(1 - \frac{\Delta\mu}{\mu} \frac{T}{\Delta T} \right) \tag{2.33}$$

$$= -\rho \frac{\Delta T}{T} \left(1 - \frac{{\rm d}\ln\mu}{{\rm d}\ln T} \right) \tag{2.34}$$

so the buoyancy force, eqtn. 2.31, is

$$f_{\rm b} = g\rho \frac{\Delta T}{T} \left(1 - \frac{{\rm d}\ln\mu}{{\rm d}\ln T} \right)$$

but force equals mass (per unit volume) times acceleration,

$$= \rho \frac{{\rm d}v}{{\rm d}t} \tag{2.35}$$

so

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g\frac{\Delta T}{T}\left(1 - \frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln T}\right) \tag{2.36}$$

where , the excess temperature of the cell (compared to the ambient gas) can be written

$$\Delta T = \left\{\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{rad}} - \left|\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{ad}}\right\} \times \Delta r \tag{2.37}$$

For constant acceleration the mean velocity over distance ℓ is given by Torricelli's equation,

$$v \simeq \sqrt{\left\{\frac{\mathrm{d}v}{\mathrm{d}t}\ell\right\}} \tag{2.38}$$

so, substituting eqtn. 2.37 for ΔT in eqtn. 2.36 (setting $\Delta r = \ell$ and neglecting a factor $\sqrt{2}$), the required velocity is

$$v = \left\{\frac{g}{T}\left|1 - \frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln T}\right|\right\}^{1/2} \left\{\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{rad}} - \left|\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{ad}}\right\}^{1/2} \times \ell \tag{2.39}$$

We can now rewrite eqtn. 2.30 as

$$F_{\mathrm{conv}} = \rho\, C_{\mathrm{P}} \left\{\frac{g}{T}\left|1 - \frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln T}\right|\right\}^{1/2} \left\{\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{rad}} - \left|\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\mathrm{ad}}\right\}^{3/2} \times \ell^2. \tag{2.40}$$

Rearranging the equation of state, eqtn. (1.2),

$$\begin{aligned}\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right| &= \frac{g\mu m(\mathrm{H})}{k}\left|\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right| \\ &= \frac{T}{H}\left|\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right| \end{aligned} \tag{2.41}$$

and so

$$F_{\mathrm{conv}} = \rho\, C_{\mathrm{P}}\, \alpha^2 T \left\{gH\left|1 - \frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln T}\right|\right\}^{1/2} \left\{\left|\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right|_{\mathrm{rad}} - \left|\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right|_{\mathrm{ad}}\right\}^{3/2} \tag{2.42}$$

which is our final formulation.

In calculating actual temperature structures in stellar envelopes, we require the total energy flux to obey

$$F = F_{\mathrm{rad}} + F_{\mathrm{conv}} = \sigma T_{\mathrm{eff}}^4 \tag{2.43}$$

The initial temperature structure is calculated on the basis of radiative transfer only ($F_{\mathrm{rad}} = \sigma T_{\mathrm{eff}}^4$), then a correction $\Delta T(r)$ computed iteratively, for given α, if the Schwarzschild criterion indicates convective transport.

Box 2.2. It may not be immediately obvious that the radiation field in stellar interiors is, essentially, isotropic; after all, outside the energy-generating core, the full stellar luminosity is transmitted across any spherical surface of radius r. However, if this flux is small compared to the local mean intensity, then isotropy is justified.

The flux at an interior radius r (outside the energy-generating core) must equal the flux at R (the surface); that is,

$$F = \sigma T_{\text{eff}}^4 \frac{R^2}{r^2}$$

while the mean intensity is

$$J_\nu(r) \simeq B_\nu(T(r)) = \sigma T^4(r).$$

Their ratio is

$$\frac{F}{J} = \left(\frac{T_{\text{eff}}}{T(r)}\right)^4 \left(\frac{R}{r}\right)^2.$$

Temperature rises rapidly below the surface of stars, so this ratio is always small; for example, in the Sun, $T(r) \simeq 3.85$ MK at $r = 0.9\text{R}_\odot$, whence $F/J \simeq 10^{-11}$. That is, the radiation field is isotropic to better than 1 part in 10^{11}.

Equivalently, the temperature gradient from the centre of the Sun (for example) to the surface is

$$\frac{\Delta T}{\Delta r} = \frac{T_{\text{c}} - T_{\text{eff}}}{R_\odot} \simeq 10^{-2} \text{ K m}^{-1} \tag{2.44}$$

The photon mean free path is $\ell = 1/\kappa \simeq 1$ mm (from detailed models), so the temperature change over this distance is of order 10^{-5} K. The radiant energy density is $U = aT^4$, so the relative anisotropy $\Delta U/U = 4\Delta T/T \simeq 10^{-11}$ at 10^6 K.

Although the anisotropy is very small, the nett outflow is large – in fact, equal to the stellar luminosity.

Section 3

Ways to make a star

Polytropes and the Lane-Emden Equation

3.1 Introduction

We've already assembled a set of equations that embody the basic principles governing stellar structure; these are

$$\frac{\mathrm{d}m(r)}{\mathrm{d}r} = 4\pi r^2 \rho(r) \qquad \text{Mass continuity;} \qquad (1.4)$$

$$\frac{\mathrm{d}P(r)}{\mathrm{d}r} = \frac{-Gm(r)\rho(r)}{r^2} \qquad \text{Hydrostatic equilibrium;} \qquad (1.1)$$

$$\frac{\mathrm{d}L(r)}{\mathrm{d}r} = 4\pi r^2\, \rho(r)\varepsilon(r) \qquad \text{Energy continuity.} \qquad (1.5)$$

Our aim is to use these to investigate (or to predict) the properties of real stars. To do this we also need descriptions of the quantities P, ε, and $\overline{k}_{\mathrm{R}}$ (pressure, energy-generation rate, and Rosseland mean opacity), which enter these equations. These are each functions (often complex functions) of density, temperature, and composition, and in modern work are computed explicitly, separately from the stellar-structure problem itself. However, progress in analytical models can be made by adopting power-law dependences,

$$\overline{k}_{\mathrm{R}}(r)/\rho(r) = \kappa(r) = \kappa_0 \rho^a(r) T^b(r) \qquad (3.1)$$

and (2112 Section 23)

$$\varepsilon(r) \simeq \varepsilon_0 \rho(r) T^\alpha(r), \qquad (3.2)$$

together with an equation of state. We'll discuss this in detail in the next section.

3.2 Polytropes

A *polytropic* equation of state is such that the pressure is assumed to be proportional to density to some power,

$$P = K\rho^{\gamma} \tag{3.3}$$

(i.e., PV^n is a constant), where
K is the polytropic constant (of proportionality),
$\gamma = (n+1)/n$ is the polytropic exponent (equivalent to the ratio of specific heats, $C_\mathrm{P}/C_\mathrm{V}$), and
n is the polytropic index (not to be confused with number-density n).
A sphere of gas satisfying eqtn. (3.3) is said to be a 'polytrope of index n'. The importance of polytropes is that they allow simple solutions to the equations of stellar structure; Eddington was able to calculate the first realistic model of the solar interior in this way. The price is the assumed decoupling of pressure from temperature; however, this turns out to be less restrictive than one might initially suppose. We'll explore this before returning to solving the equations of stellar structure

3.2.1 Convective stars as polytropes

The first law of thermodynamics states that the change in internal energy of a system, $\mathrm{d}U$, is given by the heat added to the system, $\mathrm{d}Q$, less the work done by the system, $\mathrm{d}W$:

$$\mathrm{d}U = \mathrm{d}Q - \mathrm{d}W.$$

For fully convective stars, all the convective cells are supposed to be adiabatic, so $\mathrm{d}Q \equiv 0$; and for a quasistatic process $\mathrm{d}W = P\mathrm{d}V$, whence

$$\mathrm{d}U = -P\mathrm{d}V.$$

For an ideal gas[1]

$$P = nkT \qquad = \frac{N}{V}kT$$
$$U = \frac{3}{2}NkT \qquad = \frac{3}{2}PV \qquad \text{(where } N = nV\text{)}$$

[1] A gas is close to ideal if it is fully ionized, or entirely neutral. If the gas is partially ionized, then some energy goes into ionization/dissociation, and $U \neq {}^3\!/_2 NkT$.

so that

$$
\begin{aligned}
\mathrm{d}U &= -P\mathrm{d}V; \\
\mathrm{d}\left(\frac{3}{2}PV\right) &= -P\mathrm{d}V; \\
\frac{3}{2}(P\mathrm{d}V + V\mathrm{d}P) &= -P\mathrm{d}V; \\
\frac{5}{2}P\mathrm{d}V &= -\frac{3}{2}V\mathrm{d}P; \\
\frac{\mathrm{d}P}{P} &= -\frac{5}{3}\frac{\mathrm{d}V}{V}; \\
P &\propto V^{-5/3}, \qquad \text{but } V \propto \rho^{-1} \text{ so} \\
P &\propto \rho^{5/3}. \qquad (3.4)
\end{aligned}
$$

That is, fully convective stars are approximately polytropic, with $n\,(= 1/(\gamma - 1)) = 3/2$.

or...

In a fully convective star, $P \simeq P_{\mathrm{G}}$ (i.e., radiation pressure is unimportant); then, from $P = (\rho kT)/\mu m(\mathrm{H})$, we have $T \propto (P/\rho)$.

If we have an *adiabatic* temperature gradient

$$\left(\frac{\partial T}{\partial P}\right) \equiv \nabla_{\mathrm{ad}} \rightarrow P \propto T^{1/\nabla_{\mathrm{ad}}}$$

so

$$
\begin{aligned}
P^{1-(1/\nabla_{\mathrm{ad}})} &\propto \rho^{-1/\nabla_{\mathrm{ad}}}, \text{ or} \\
P &= K\rho^{1/(1-\nabla_{\mathrm{ad}})}
\end{aligned}
$$

(and for a perfect gas, $P \propto \rho^{5/3}$).

3.2.2 Radiative stars as polytropes: the Eddington Model

The *total* pressure P is the sum of gas pressure P_{G} and radiation pressure P_{R},[2] where

$$
\begin{aligned}
P_{\mathrm{G}} &= \frac{\rho}{\mu m(\mathrm{H})}kT, \qquad \text{and} \qquad (1.2) \\
P_{\mathrm{R}} &= \frac{1}{3}aT^4. \qquad (1.3)
\end{aligned}
$$

We define

$$
\begin{aligned}
P_{\mathrm{R}} &\equiv (1 - \beta_{\mathrm{P}})P \qquad (3.5) \\
\text{i.e.,} \quad \frac{P_{\mathrm{G}}}{P_{\mathrm{R}}} &= \frac{\beta_{\mathrm{P}}}{1 - \beta_{\mathrm{P}}}.
\end{aligned}
$$

[2]Plus degeneracy pressure, which is negligible fornormal stars.

We can therefore write

$$\begin{aligned} P &= \frac{P_{\rm G}}{\beta_{\rm P}} \\ &= \frac{\rho k T}{\beta_{\rm P} \mu m({\rm H})}; \end{aligned}$$

rearranging for T, substituting the result into eqtn. (1.3) and inserting the result into eqtn. (3.5) gives

$$P^4 \left(\frac{\beta_{\rm P} \mu m({\rm H})}{\rho k} \right)^4 = \frac{3(1-\beta_{\rm P})}{a} P$$

i.e.,

$$\begin{aligned} P &= \left\{ \frac{k}{\mu m({\rm H})} \right\}^{4/3} \left\{ \frac{3(1-\beta_{\rm P})}{a \beta_{\rm P}^4} \right\}^{1/3} \rho^{4/3}, \\ &\equiv K \rho^{4/3} \qquad \text{if } \beta_{\rm P} \text{ (and } \mu\text{) are constant.} \end{aligned} \tag{3.6}$$

That is, for constant $\beta_{\rm P}$ and constant μ, $P \propto \rho^{4/3}$ – a polytropic equation of state with polytropic index $n = 3$. The behaviour of $n = 3$ polytropes was investigated by Eddington, and is embodied in the *'Eddington Standard Model'*.

The assumption of constant μ is likely to be reasonable, at least for main-sequence stars, but is constant $\beta_{\rm P}$ plausible? We know that radiative energy transport is described by

$$\frac{{\rm d}T}{{\rm d}P} \equiv \frac{{\rm d}T}{{\rm d}P_{\rm R}} \frac{{\rm d}P_{\rm R}}{{\rm d}P} = -\frac{3 \overline{\kappa}_{\rm R} L(r)}{16\pi\, acT^3\, Gm(r)}. \tag{2.11}$$

Since $P_{\rm R} = \frac{1}{3} a T^4$ (eqtn.1.3) we have

$$\frac{{\rm d}T}{{\rm d}P_{\rm R}} = \frac{3}{4aT^3},$$

whence

$$\frac{{\rm d}P_{\rm R}}{{\rm d}P} = \frac{\overline{\kappa}_{\rm R} L(r)}{4\pi c G m(r)}.$$

Integrating gives[3]

$$\frac{P_{\rm R}}{P} \simeq \frac{\overline{\kappa}_{\rm R} L}{4\pi c G M}.$$

assuming negligible surface pressures and constant $\overline{\kappa}_{\rm R}$ (not unreasonable for hot stars). That is, for stars where energy transport is radiative, there is a constant ratio of gas pressure to radiation pressure (i.e., constant $\beta_{\rm P}$), and so the Eddington model is indeed applicable.

[3]The astute student may note the connection with the Eddington limit to luminosity.

3.2.3 Degenerate stars as polytropes: white dwarfs

White dwarfs are supported by electron degeneracy pressure (or 'Fermi pressure'), so we anticipate an equation of state that is independent of temperature. The uncertainty principle tells us that, in general,

$$\Delta p \Delta x \geq \hbar/2$$

where p is the momentum and x is some spatial co-ordinate. The electron momentum must be of the order of the uncertainty, i.e.,

$$p_{\rm e} \simeq \hbar/\Delta x.$$

If the electron density is $n_{\rm e}$ then the average spacing between electrons is of order

$$\Delta x \sim n_{\rm e}^{-1/3}, \qquad \text{so}$$
$$p_{\rm e} \sim \hbar n_{\rm e}^{1/3}$$

but, in the non-relativistic limit, $p_{\rm e} = m_{\rm e} v$; i.e.,

$$v \sim n_{\rm e}^{1/3}.$$

Finally, the electron degeneracy pressure (momentum per unit area per unit time) is

$$P_{\rm e} \sim n_{\rm e} \cdot v \;\; p; \qquad \text{that is,}$$
$$P_{\rm e} \propto n_{\rm e}^{5/3}, \quad \propto \rho^{5/3}.$$

Detailed calculations give the constant of proportionality.

$$\begin{aligned} P_{\rm e,n} &= \frac{h^2}{20 m_{\rm e} m({\rm H}) \mu_{\rm e}} \left(\frac{3}{\pi m({\rm H}) \mu_{\rm e}} \right)^{2/3} \rho^{5/3} \\ &\equiv K_1 \rho^{5/3} \end{aligned} \tag{3.7}$$

(where $\mu_{\rm e}$ is the mean number of nucleons per electron), showing that non-relativistic white dwarfs are well approximated by polytropes with $P \propto \rho^{5/3} (n = 3/2)$.

In the relativistic limit $v \simeq c$, so

$$P_{\rm e} \sim n_{\rm e} \cdot c \;\; p; \qquad \text{i.e.,}$$
$$P_{\rm e} \propto \rho^{4/3},$$

or in detail

$$\begin{aligned} P_{\rm e,r} &= \frac{hc}{8 m({\rm H}) \mu_{\rm e}} \left(\frac{3}{\pi m({\rm H}) \mu_{\rm e}} \right)^{1/3} \rho^{4/3}, \\ &\equiv K_2 \rho^{4/3} \end{aligned} \tag{3.8}$$

so that relativistic white dwarfs are also polytropic, though with a different polytropic index ($n = 3$).

[The same general arguments apply to neutron stars, though with different K coefficients.]

3.2.4 Summary of ∼polytropic stars

To summarize, the following systems are reasonably approximated as polytropes:
– convective stars

$$P \propto \rho^{5/3} \qquad n = 1.5. \tag{3.4}$$

– stars with constant $\beta_{\rm P}$) (∼radiative stars)

$$P \propto \rho^{4/3} \qquad n = 3; \tag{3.6}$$

– non-relativistic white dwarfs,

$$P_{\rm e,n} \propto \rho^{5/3} \qquad n = 1.5; \tag{3.7}$$

– relativistic white dwarfs,

$$P_{\rm e,r} \propto \rho^{4/3} \qquad n = 3; \tag{3.8}$$

3.3 The Lane-Emden Equation

We can now proceed with using results we've assembled so far to investigate stellar structure, using polytropic models. The *Lane-Emden equation* represents a solution of the equations of stellar structure for such models, expressed in a dimensionless form.

We start with the 'mechanical' equations of hydrostatic equilibrium and mass continuity,

$$\frac{{\rm d}P(r)}{{\rm d}r} = -\frac{Gm(r)\rho(r)}{r^2} \tag{1.1}$$

$$\frac{{\rm d}m}{{\rm d}r} = 4\pi r^2 \rho(r). \tag{1.4}$$

Rearranging and differentiating eqtn. (1.1) gives us

$$\frac{{\rm d}}{{\rm d}r}\left(\frac{r^2}{\rho}\frac{{\rm d}P}{{\rm d}r}\right) = -G\frac{{\rm d}m(r)}{{\rm d}r}$$

whence, from eqtn. (1.4),

$$\frac{1}{r^2}\frac{{\rm d}}{{\rm d}r}\left(\frac{r^2}{\rho}\frac{{\rm d}P}{{\rm d}r}\right) = -4\pi G\rho \tag{3.9}$$

This already embodies the essential aspects of the Lane-Emden equation.[4] It describes (in principle) how P, ρ vary with radius r in a star. We need to eliminate one or other of these, which amounts to determining T as a function of r.

Therefore, having dealt with 'mechanical' issues, we now need to turn to the thermal structure. It might appear that we can't solve hydrostatic equilibrium without knowing something about the pressure, i.e., the temperature, which in turn suggests needing to know about energy generation processes, opacities, and other complexities. Surprisingly, however, we can find interesting results without these details, under some not-too-restrictive assumptions about the equation of state. Specifically, we adopt our (temperature-independent!) polytropic law,

$$P = K\rho^{\gamma} = K\rho^{(n+1)/n} \tag{3.3}$$

where we recall that K is a constant, γ is the ratio of specific heats,[5] and n is the polytropic index. In adopting this form, we see the virtue of polytropes – eqtn. (3.3) is, of course, independent of temperature.

Introducing eqtn. (3.3) into (3.9) gives

$$\frac{K}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}\rho^{\gamma}}{\mathrm{d}r}\right) = -4\pi G\rho. \tag{3.10}$$

At least in principle, we can now solve for $\rho(r)$, given some boundary conditions; if we wanted to, we'd now do this by some numerical computational method. Before the era of electronic computers, this wasn't feasible, so the pioneer investigators of stellar structure proceeded to simplify eqtn. (3.10) by introducing dimensionless scaled variables in place of the physical variables r and T:

$$\xi = \frac{r}{\alpha}, \tag{3.11}$$

$$\theta = \frac{T}{T_{\mathrm{c}}} \tag{3.12}$$

[4]It's also, essentially, Poisson's equation; from hydrostatic equilibrium,

$$\frac{1}{\rho}\frac{\mathrm{d}P(r)}{\mathrm{d}r} = -g(r) = \frac{\mathrm{d}\psi}{\mathrm{d}r}$$

whence

$$\nabla^2\psi = 4\pi G\rho$$

in spherical coördinates.

[5]This relation need not necessarily be taken to be an equation of state – it simply expresses an assumption regarding the evolution of pressure with radius, in terms of the evolution of density with radius. The Lane-Emden equation has applicability outside stellar structures.

(where α is a constant with units of length, and T_c is the core temperature). The scaled density follows from the so-called Emden transformation and to the polytropic equation of state (eqtn. (3.3):

$$\begin{aligned}
\frac{\rho}{\rho_c} &= \left(\frac{P}{P_c}\right)^{1/\gamma} = \left(\frac{\rho T}{\rho_c T_c}\right)^{1/\gamma} \text{; i.e.,} \\
\left(\frac{\rho}{\rho_c}\right)^{(\gamma-1)/\gamma} &= \left(\frac{T}{T_c}\right)^{1/\gamma} = \theta^{1/\gamma}, \text{ or} \\
\frac{\rho}{\rho_c} &= \theta^{1/(\gamma-1)}, \quad = \theta^n
\end{aligned} \tag{3.13}$$

(where we have also used $P \propto \rho T$). We can therefore think of θ as being a measure of either temperature (eqtn. 3.12) or density (eqtn. 3.13).

Then eqtn. (3.10) becomes

$$\frac{K}{(\alpha\xi)^2}\frac{\mathrm{d}}{\mathrm{d}(\alpha\xi)}\left\{\frac{(\alpha\xi)^2}{\rho_c\theta^n}\frac{\mathrm{d}\left(\{\rho_c\theta^n\}^{(n+1)/n}\right)}{\mathrm{d}(\alpha\xi)}\right\} = -4\pi G\rho_c\theta^n.$$

This looks pretty intimidating, but since

$$\frac{\mathrm{d}}{\mathrm{d}\xi}(\theta^{n+1}) = (n+1)\theta^n\frac{\mathrm{d}\theta}{\mathrm{d}\xi}$$

this reduces to

$$\frac{(n+1)K\rho_c^{(1-n)/n}}{4\pi G\alpha^2}\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n.$$

This is still 'symbol soup', but letting the constant α (which is freely selectable, provided its dimensionality – length – is preserved) be

$$\alpha \equiv \left\{\frac{(n+1)K\rho_c^{(1-n)/n}}{4\pi G}\right\}^{1/2}, \tag{3.14}$$

we obtain a compact second-order differential equation relating (scaled) radius to (scaled) temperature or, equivalently (scaled) density:

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n \left[= -\frac{\rho}{\rho_c}\right] \tag{3.15}$$

which is the standard form of the Lane-Emden equation.[6,7]

---

[6]If desired, we can expand this to

$$\frac{1}{\xi^2}\left(2\xi\frac{\mathrm{d}\theta}{\mathrm{d}\xi} + \xi^2\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2}\right) + \theta^n = \frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} + \frac{2}{\xi}\frac{\mathrm{d}\theta}{\mathrm{d}\xi} + \theta^n = 0.$$

[7]The essential physics entering this derivation was described by the American astronomer Jonathan Lane in 1870; the equation, in this form, was published by the Swiss Jacob Robert Emden (who was Karl Schwarzschild's brother-in-law) in 1907, although it was first given by Ritter in 1880.

3.3.1 Solutions

The Lane-Emden equation is a dimensionless form of eqtn. (3.10), and its solutions give us, essentially, density (and other quantities) as a function of radius, for appropriate boundary conditions. Because it is a second-order differential equation, we need two boundary conditions.

A first is obtained trivially from the definition of θ;

$$\theta(\xi = 0) = 1;$$

i.e., $T = T_\mathrm{c}$ (or $\rho = \rho_\mathrm{c}$) at $r = 0$. Since $\mathrm{d}P/\mathrm{d}r \to 0$ as $r \to 0$, we also have

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right|_{\xi=0} = 0$$

at $\xi = 0$.

Numerical solutions are fairly straightforward to compute with these boundary conditions; analytical solutions are possible for polytropic indexes $n = 0$, 1 and 5 (i.e., ratios of specific heats $\gamma = \infty, 2$, and 1.2); these are, respectively,[8]

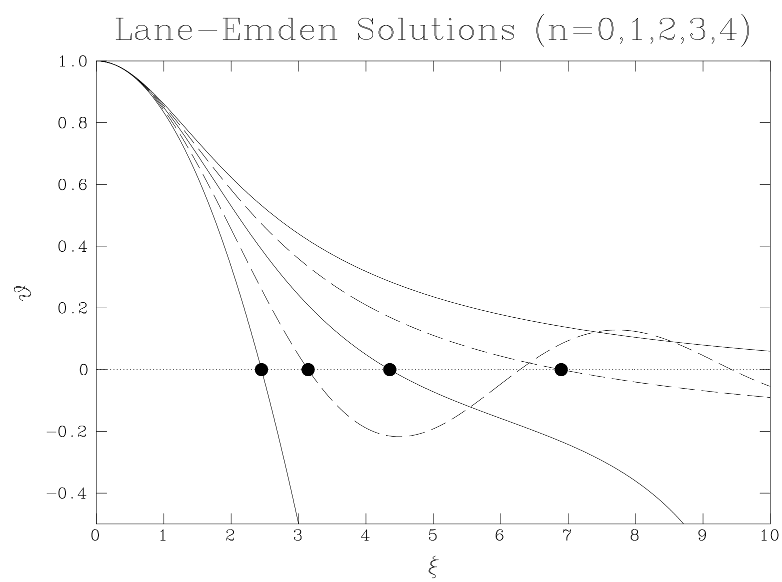
$$\begin{aligned} \theta(\xi) &= 1 - \xi^2/6 & \xi_1 &= \sqrt{6}, \\ &= \sin\xi/\xi & &= \pi, \text{ and} \\ &= \left(1 + \xi^2/3\right)^{-1/2} & &= \infty \end{aligned}$$

where ξ_1 is the first root of θ (i.e., the smallest positive value of ξ for which $\theta = 0$) – that is, the rescaled radius of the star, R/α. 'Solutions' of the Lane-Emden equation are very often expressed solely in terms of surface ('ξ_1') values of ξ and $\mathrm{d}\theta/\mathrm{d}\xi$ (or $-\xi^2\,\mathrm{d}\theta/\mathrm{d}\xi$), as in the following table (which I calculated using a very simple numerical integration):

[8] Details of the solutions are available at `http://mathworld.wolfram.com/Lane-EmdenDifferentialEquation.html`

Solutions of the Lane-Emden Equation

n	ξ_1	$-\xi_1^2 \left.\frac{d\theta}{d\xi}\right\|_{\xi_1}$
0.0	2.4495	4.8990
0.5	2.7527	3.7887
1.0	3.1416	3.1416
1.5	3.6538	2.7141
2.0	4.3529	2.4111
2.5	5.3553	2.1872
3.0	6.8968	2.0182
3.5	9.5358	1.8906
4.0	14.972	1.7972
4.5	31.836	1.7378
5.0	∞	1.7321



Solutions of the Lane-Emden equation for several values of the polytropic index n (increasing left to right).

A polytrope with index $n = 0$ has a uniform density, while a polytrope with index $n = 5$ has an infinite radius. In general, the larger the polytropic index, the more centrally condensed the density distribution; and only polytropes with $n \leq 5$ are bound systems (Section 3.5). [A polytrope with index $n = \infty$ corresponds to a so-called 'isothermal sphere', a self-gravitating, isothermal gas sphere, used to analyse collisionless systems of stars (in particular, globular clusters).]

3.4 Astrophysical Solutions

We've done a lot of algebra; what about the physical interpretation of all this? Recall that the Lan-Emden equation is the solution of the equations of hydrostatic equilibrium and mass continuity, for a polytropic equation of state, expressed in dimensionless form.

That is, the solution of Lane-Emden equation depends only on the polytropic index, n, but is expressed in terms of *scaled* parameters. If we want to obtain astrophysically more interesting solutions (in terms of actual stellar masses, radii, etc.) then we need two physical parameters to transform the scaled ones. We might choose actual numerical values for K and $\rho_{\rm c}$, for example, but other pairings (e.g., stellar mass and radius) would serve. Under these circumstances, we can then evaluate:

- The stellar radius,

$$R \equiv \alpha\xi_1 \tag{3.11}$$

$$= \left\{\frac{K(n+1)}{4\pi G}\right\}^{1/2} \rho_{\rm c}^{(1-n)/(2n)}\xi_1 \tag{3.16}$$

(from eqtn. 3.14).

- The stellar mass,

$$M = \int_0^R 4\pi r^2 \rho {\rm d}r$$

but $r = \alpha\xi$ (eqtn. 3.11) and $\rho = \rho_{\rm c}\theta^n$ (eqtn. 3.13), so

$$M = 4\pi\alpha^3\rho_{\rm c}\int_0^{\xi_1} \xi^2\theta^n {\rm d}\xi;$$

then using the Lane-Emden equation, (3.15), for θ^n,

$$M = -4\pi\alpha^3\rho_{\rm c}\int_0^{\xi_1} \frac{\rm d}{{\rm d}\xi}\left(\xi^2\frac{{\rm d}\theta}{{\rm d}\xi}\right){\rm d}\xi$$

$$= 4\pi\left\{\frac{K(n+1)}{4\pi G}\right\}^{3/2} \rho_{\rm c}^{(3-n)/(2n)} \left\{-\xi^2\frac{{\rm d}\theta}{{\rm d}\xi}\bigg|_{\xi_1}\right\} \tag{3.17}$$

- The stellar density,

$$\overline{\rho} = \frac{3M}{4\pi R^3}$$

$$= \rho_{\rm c}\frac{3}{\xi_1^3}\left\{-\xi^2\frac{{\rm d}\theta}{{\rm d}\xi}\bigg|_{\xi_1}\right\}, \tag{3.18}$$

(where we use our previous results, eqtns. (3.17) and (3.16) for M and R).

- The central pressure can be expressed trivially as

$$P_c = K\rho_c^{(n+1)/n}, \tag{3.3}$$

or, in terms of the mass and radius,

$$= \left\{ 4\pi(n+1) \left(\left.\frac{d\theta}{d\xi}\right|_{\xi_1} \right)^2 \right\}^{-1} \frac{GM^2}{R^4} \tag{3.19}$$

(obtained by eliminating ρ_c between eqtns (3.16) and (3.17) to obtain K in terms of M and R, and using eqtn. (3.18) for ρ_c).

3.4.1 Application to white dwarfs: mass–radius relation

The central density, ρ_c, can be eliminated between equations (3.16) and (3.17) to give

$$R^{(3-n)}M^{(n-1)} = \left\{ 4\pi \left(-\xi^2 \left.\frac{d\theta}{d\xi}\right|_{\xi_1} \right) \right\}^{(n-1)} \left\{ \frac{K}{G}\frac{n+1}{4\pi} \right\}^n \xi_1^{3-n}. \tag{3.20}$$

The right-hand side is a (dimensionless) number that depends only on the polytropic index. We see that polytropes therefore follow a mass–radius relationship,

$$M \propto R^{(n-3)/(n-1)}. \tag{3.21}$$

We know that non-relativistic white dwarfs are polytropes with $P \propto \rho^{5/3}$ (i.e., $n = 3/2$; Section 3.2.3), whence $M \propto R^{-3}$. Using eqtn. (3.7) in eqtn. (3.20), and inserting numerical values, we obtain a quantitative mass-radius relation for non-relativistic white dwarfs,

$$R = \frac{K_1}{0.4242GM^{1/3}}. \tag{3.22}$$

Evaluating the numerical factors, we find

$$\frac{R}{R_\odot} \simeq 10^{-2} \frac{M}{M_\odot}^{-1/3},$$

showing that when a star like our Sun becomes a white dwarf it will have about the same radius as that of the Earth (∼6000 km),and hence a density of $\sim 10^9$ kg m^{-3}.

With increasing mass, the radius decreases, and the density rises, eventually entering the relativistic regime. (As the particles get squeezed into smaller and smaller volumes, the uncertainty principle implies that the velocities are larger and larger.) Then

$$P = P_{e,r} = K_2\rho^{4/3},$$

where

$$K_2 = \frac{h^2}{20 m_e m(\mathrm{H}) \mu_e} \left(\frac{3}{\pi m(\mathrm{H}) \mu_e} \right)^{2/3}, \tag{3.8}$$

Again inserting numerical values

$$\begin{aligned} M &= \left(\frac{K_2}{0.3639 G} \right)^{3/2} \\ &= 1.142 \times 10^{31} \mu_e^{-2} \text{ kg}, \\ &= 5.74 \mu_e^{-2} \mathrm{M}_\odot \end{aligned}$$

This is the maximum mass that can be supported by electron degeneracy pressure; for a helium white dwarf, $\mu_e \simeq 2$, and this 'Chandrasekhar mass' is $M_{\mathrm{Ch}} \simeq 1.44\mathrm{M}_\odot$. A star more massive than the Chandraskhar limit that has no other significant sources of pressure will continue to collapse, to a neutron star (supported by neutron degeneracy pressure) or black hole.[9]

3.4.2 The Eddington model

Some results for the Eddington model ($n = 3$, constant β) are set in Problem Sheet 1. We note one additional result, namely that from eqtn. (3.6) we have

$$K \propto \left\{ \frac{(1 - \beta_{\mathrm{P}})}{\beta_{\mathrm{P}}^4} \right\}^{1/3}$$

(where $P_{\mathrm{R}} = (1 - \beta)P$, while from eqtn. (3.17), with $n = 3$

$$K \propto M^{2/3};$$

that is,

$$\left\{ \frac{(1 - \beta_{\mathrm{P}})}{\beta_{\mathrm{P}}^4} \right\} \propto M^2.$$

This shows us that as M increases, β_{P} decreases (the left-hand side is a monotonically decreasing function of β_{P}) – that is, with increasing mass, radiation pressure becomes increasingly important. While stars like the Sun are largely supported by gas pressure, the most massive stars are almost entirely supported by radiation.

[9]The equation of state for nuclear matter is not well established, so that the upper limit for a neutron-star mass – the 'Tolman–Oppenheimer–Volkoff limit' – is less certain than is the Chandrasekhar limit. The most accurate determinations of neutron-star masses (from binary-star systems) are all remarkably close to $1.4\mathrm{M}_\odot$, but with a range of $\sim$1–$2\mathrm{M}_\odot$.

3.4.3 An alternative look at the mass-radius relation: white dwarfs [not for lectures]

The kinetic energy of a particle of mass m, velocity v, is

$$E_{\rm p} = \frac{1}{2}mv^2;$$

and the uncertainty principle tells us that

$$\Delta x \Delta p \geq \hbar/2$$

where p is the momentum and x is some spatial co-ordinate such that

$$\Delta x^3 = V/N$$

for a star of volume V containing N particles. That is, the uncertainty principle shows us that

$$p^2 \propto \hbar^2 N^{2/3}/R^2$$

whence the total kinetic energy of the entire star ($\propto N$) is

$$E_* \propto (\hbar^2 N^{5/3})/(mR^2)$$

The star is in equilibrium when the kinetic energy equals the gravitational energy; that is, when

$$(\hbar^2 N^{5/3})/(mR^2) \propto M^{5/3}/R^2 \simeq GM^2/R,$$

or

$$R \propto M^{-1/3}$$

as before.

As the mass increases the radius gets smaller, so we must eventually enter the relativistic regime (Δx decreases so Δp increases). The relativistic form for the energy in the limit $v \to c$ is

$$E_{\rm p} = pc;$$

then repeating the arguments from above leads to

$$M^{2/3} \approx \hbar c m_{\rm p}^{-4/3}/G \simeq 1.4{\rm M}_\odot.$$

That is, *the maximum mass of a star that can be supported by electron degeneracy pressure depends only on physical constants,* and not on any other property of the star.

3.5 Binding energy [not for lectures]

It's of interest also to investigate the total energy of a polytrope. We start by considering the gravitional potential energy; this is given in the usual way by

$$\begin{aligned}\Omega &= -\int_{\mathrm{c}}^{\mathrm{s}} \frac{Gm}{r}\,\mathrm{d}m \\ &\equiv -\int_{\mathrm{c}}^{\mathrm{s}} \frac{G}{2r}\,\mathrm{d}(m^2)\end{aligned} \tag{3.23}$$

(where the c, s limits mean 'centre' and 'surface'). Integrating by parts,

$$\Omega = -\frac{GM^2}{2R} - \int_{\mathrm{c}}^{\mathrm{s}} \frac{Gm^2}{2r^2}\,\mathrm{d}r,$$

but from hydrostatic equlibrium, $\mathrm{d}r = -(r^2 \mathrm{d}P)/(Gm\rho)$, so this can be written as

$$\Omega = -\frac{GM^2}{2R} + \int_{\mathrm{c}}^{\mathrm{s}} \frac{\cancel{Gm}m}{2\cancel{r^2}}\,\frac{\cancel{r^2}}{\cancel{Gm}}\,\frac{\mathrm{d}P}{\rho}. \tag{3.24}$$

We now use the polytropic equation of state,

$$P = K\rho^{\gamma} = K\rho^{(n+1)/n}, \tag{3.3}$$

to obtain

$$\frac{\mathrm{d}P}{\mathrm{d}\rho} = K\frac{n+1}{n}\rho^{1/n};$$

we also use the trick of writing eqtn. (3.3) as

$$\frac{P}{\rho} = K\rho^{1/n}$$

to write

$$\begin{aligned}\frac{\mathrm{d}\,(P/\rho)}{\mathrm{d}\rho} &= \frac{K}{n}\rho^{(1-n)/n} \\ \frac{\mathrm{d}\,(P/\rho)}{\cancel{\mathrm{d}\rho}} &= \frac{\mathrm{d}P}{\rho\cancel{\mathrm{d}\rho}}\frac{1}{n+1}; \qquad \text{i.e.,} \\ \frac{\mathrm{d}P}{\rho} &= (n+1)\,\mathrm{d}\,(P/\rho)\end{aligned} \tag{3.25}$$

Substituting back into eqtn. (3.24) gives us

$$\Omega = -\frac{GM^2}{2R} + \frac{n+1}{2}\int_{\mathrm{c}}^{\mathrm{s}} m\,\mathrm{d}\,(P/\rho); \tag{3.26}$$

integrating by parts for a second time,

$$= -\frac{GM^2}{2R} + \left[\frac{n+1}{2}m\frac{P}{\rho}\right]_{\mathrm{c}}^{\mathrm{s}} - \frac{n+1}{2}\int_{\mathrm{c}}^{\mathrm{s}} \frac{P}{\rho}\,\mathrm{d}m.$$

The term in [brackets] vanishes ($m = 0$ at the centre, $P/\rho = 0$ at the surface), so, using mass continuity (eqtn. 1.4),

$$\Omega = -\frac{GM^2}{2R} - \frac{n+1}{2}\int_{\rm c}^{\rm s} \frac{P}{\rho} 4\pi r^2 \rho \,\mathrm{d}r \tag{3.27}$$

$$= -\frac{GM^2}{2R} - \frac{n+1}{2}\int_{\rm c}^{\rm s} P\frac{4\pi}{3}\,\mathrm{d}r^3. \tag{3.28}$$

Integrating by parts yet again,

$$\Omega = -\frac{GM^2}{2R} - \left[\frac{n+1}{2}\frac{4\pi}{3}Pr^3\right]_{\rm c}^{\rm s} + \frac{n+1}{6}\int_{\rm c}^{\rm s} 4\pi r^3\,\mathrm{d}P. \tag{3.29}$$

The second term again vanishes $r_{\rm c} = 0$, $P_{\rm s} = 0$); then, using the equation of hydrostatic equilibrium once more,

$$\Omega = -\frac{GM^2}{2R} - \frac{n+1}{6}\int_{\rm c}^{\rm s} 4\pi r^3 \frac{Gm\rho}{r^2}\,\mathrm{d}r, \tag{3.30}$$

and mass continuity,

$$\Omega = -\frac{GM^2}{2R} - \frac{n+1}{6}\int_{\rm c}^{\rm s} \frac{Gm}{r}\,\mathrm{d}m \tag{3.31}$$

– but the integral is the definition of Ω (eqtn. 3.23) that we started with! That is,

$$\Omega = -\frac{GM^2}{2R} - \frac{n+1}{6}\Omega, \qquad \text{or} \tag{3.32}$$

$$\Omega = \left(\frac{3}{n-5}\right)\frac{GM^2}{R}. \tag{3.33}$$

We can then appeal to the virial theorem to obtain the total energy,

$$E = \Omega + U = \frac{1}{2}\Omega = \left(\frac{3}{n-5}\right)\frac{GM^2}{2R}.$$

We see that for $n > 5$ the energy is $E > 0$, meaning the system is unbound; only polytropes with $n < 5$ are gravitationally bound, and hence potentially of interest as stellar models.

Section 4

Pre-main-sequence evolution: polytropic models

4.1 Introduction: review

We now turn to evolution of stars. We'll divide this into three broad sections, namely
– *Pre-main-sequence evolution;*
– *Main-sequence evolution;* and
– *Post-main-sequence evolution.*

For reference, we'll remind ourselves of the Virial theorem, and some relevant timescales (from PHAS2112) that will be of use:

- *Virial Theorem* The virial theorem expresses the relationship between the gravitational energy Ω and thermal energy U in a 'virialised system' (such as a star):

  $$2U + \Omega = 0 \tag{4.1}$$

  or $U = -\Omega/2$; thus the total energy is

  $$E = U + \Omega = \Omega/2,$$

  (which must be negative for the star to be bound).

- The *dynamical timescale,* on which departures from hydrostatic equilibrium are restored in stellar systems,

  $$t_{\rm dyn} \simeq \sqrt{\frac{r^3}{Gm(r)}} \tag{4.2}$$
  $$\simeq 1/\sqrt{(G\rho)} \qquad \sim 2000 \text{ s for the Sun}$$

- The *Kelvin-Helmholtz timescale,* the time taken for the gravitational potential energy to be radiated away:

  $$t_{\rm KH} = \frac{|\Omega|}{L}$$

  where

  $$\begin{aligned}\Omega &= \int_0^R -G\frac{16\pi^2}{3} r^4 \overline{\rho}^2(r)\,{\rm d}r \\ &\simeq -\frac{16}{15}\pi^2 G\overline{\rho}^2 R^5 \\ &= -\frac{9}{15}\frac{GM^2}{R}\end{aligned}$$

  (assuming $\overline{\rho}(r) = \overline{\rho}(R)$). The Kelvin-Helmholtz timescale for the Sun ($\overline{\rho} = 1.4 \times 10^3$ kg m^{-3}, $\Omega = 2.2 \times 10^{41}$ J) is $t_{\rm KH} \simeq 10^7$ yr.

- The *thermal timescale,* the time taken for thermal energy to be radiated away, is a factor 2 smaller (through the Virial theorem), but is of the same order of magnitude; i.e.,

  $$t_{\rm th} \simeq t_{\rm KH}$$

- The *nuclear timescale* is a measure of how long it takes the reservoir of nuclear energy to be released,

  $$t_{\rm N} = \frac{f_{\rm N} M c^2}{L} \tag{4.3}$$

  where $f_{\rm N}$ is just the fraction of the rest mass available to the relevant nuclear process. In the case of hydrogen burning the fractional 'mass defect' is 0.007, so we might expect

  $$t_{\rm N} = \frac{0.007{\rm M}_\odot c^2}{{\rm L}_\odot} (\simeq 10^{11} \text{ yr for the Sun}).$$

  However, in practice, only the core of the Sun – about ∼10% of its mass – takes part in hydrogen burning, so $f_{\rm N} \simeq 10^{-3}$, and the nuclear timescale for hydrogen burning is $\sim 10^{10}$ yr for the Sun. Other evolutionary stages have their respective (shorter) timescales.

Overall, $t_{\rm dyn} < t_{\rm KH} \simeq t_{\rm th} < t_{\rm N}$. We will see that each of these timescales is appropriate, in turn, in the evolution of stars.

4.2 Jeans mass

Observationally, we know that stars form from interstellar matter, typically in groups (clusters and associations). The essential physical process is evidently one of gravitational collapse, which occurs on a dynamical timescale. However, because the densities are low, even in the 'dense' clouds associated with star formation ($n \sim 10^4$ cm^{-3}, $\sim 10^{-17} kg$ m^{-3}), even this timescale is quite long ($\sim 10^6$ yr).

We'll consider collapse from a simple, idealised perspective. The thermal energy of our initial system is

$$U = \frac{3}{2}NkT$$

where N is the number of particles, nV. The gravitational potential energy is

$$\Omega = -f\frac{GM^2}{R}$$

where f is a factor, of order unity, that depends on the mass distribution; for uniform density, $f = 3/5$, but more centrally condensed systems yield larger values of f. So, the Virial theorem for an initially uniform gas cloud becomes

$$3NkT = \frac{3}{5}\frac{GM^2}{R}$$

If the equality actually holds, then the system is in virial equilibrium (and nothing much happens). If the left-hand side is larger, thermal energy exceeds the gravitational potential energy, and the cloud disperses. If the right-hand side is greater, gravity wins, and the cloud collapses to a protostar. Thus our condition for collapse is

$$3NkT < \frac{3}{5}\frac{GM^2}{R}. \tag{4.4}$$

For mean density $\overline{\rho}$ and cloud mass M, the corresponding characteristic length scale is

$$R = \left(\frac{3M}{4\pi\overline{\rho}}\right)^{1/3}$$

(smaller clouds collapse) and, substituting $N = M/(\mu m(\mathrm{H}))$, eqtn. (4.4) can be written as

$$M > \left(\frac{5kT}{G\mu m(\mathrm{H})}\right)^{3/2}\left(\frac{3}{4\pi\overline{\rho}}\right)^{1/2} \simeq \left(\frac{kT}{G\mu m(\mathrm{H})}\right)^{3/2}\overline{\rho}^{-1/2} \tag{4.5}$$

(to order of magnitude); i.e., $M_\mathrm{J} \propto T^{3/2}\rho^{-1/2}$. This is the constraint on cloud mass required for collapse. This sort of argument was first put forward by Sir James Jeans, and this critical mass is therefore known as the *Jeans mass,* M_J. We see that the easiest clouds to collapse are cold, dense ones (typically, dense molecular clouds).

[Jeans himself derived this limiting mass by supposing that a cloud collapses if the sound-crossing time is greater than the free-fall time. Jeans' original argument was actually flawed, but his general results still provides a useful rule of thumb indicating whether or not a given system is liable to collapse.]

We can also arrange eqtn. (4.4) to solve for R,

$$R = \left(\frac{15}{8\pi} \frac{kT}{G\mu m(\mathrm{H})\overline{\rho}} \right)^{1/2}, \tag{4.6}$$

the *Jeans length* – larger clouds (at the specified density and temperature) will collapse.

What actually causes a molecular cloud to collapse? Observationally, star formation is observed to be triggered by density waves in spiral galaxies, or sequentially by (presumably) supernova blast waves.

4.3 Protostars: contraction

The Virial Theorem tells us how a virialised system responds to changing conditions, namely

$$\Delta U = -\frac{\Delta\Omega}{2}.$$

Thus as a gas cloud contracts it gets hotter (U becomes more positive as Ω becomes more negative; that is, as the system becomes more tightly bound). Only *half* the change in Ω has been accounted for; the remaining energy is 'lost' – in the form of radiation.

As the temperature rises, first molecules (principally H_2) dissociate, then ionization of hydrogen and helium occurs. Eventually, hydrostatic equilibrium is established as a result of rising pressure, and the collapsing gaseous condensation has become a *protostar*.

We can therefore *roughly* estimate the properties of the protostar by supposing that all the available gravitational potential energy initially released in collapse (from infinity to some protostellar radius $R_{\rm ps}$) is used in dissociation and ionization; that is, that

$$\frac{GM^2}{R_{\rm ps}} \simeq \frac{M}{m(\mathrm{H})} \left(\frac{X}{2} \cdot \chi(\mathrm{H_2}) + X \cdot \chi(\mathrm{H}) + \frac{Y}{4} \cdot \chi(\mathrm{He}) \right) \tag{4.7}$$

(neglecting some factors of order unity), where
$X, Y (\simeq 1 - X)$ are the abundances by mass of hydrogen and helium,
$\chi(\mathrm{H_2})$ is the dissociation energy of molecular hydrogen (4.5 eV), and
$\chi(\mathrm{H}), \chi(\mathrm{He})$ are the ionization potentials of hydrogen and helium (13.6 eV, and 24.6+54.4 eV), giving

$$\begin{aligned} \frac{R_{\rm ps}}{\mathrm{R}_\odot} &\simeq \frac{43}{(1 - 0.2X)} \frac{M}{\mathrm{M}_\odot} \\ &\sim 50 \text{ for a solar-mass star.} \end{aligned} \tag{4.8}$$

In reality, additional energy is lost through other mechanisms (outflows etc.), and this 'back of the envelope' calculation significantly overestimates $R_{\rm ps}$. Nevertheless, it provides us with a simple, if crude, estimate of the maximum radius for a protostar as it begins its evolution. We can also estimate the average internal temperature, from the Virial theorem,

$$\begin{aligned}\overline{T} &\simeq \frac{\mu m(\mathrm{H})}{3k}\frac{GM}{R_{\mathrm{ps}}} \\ &\sim 10^5\ \mathrm{K}.\end{aligned}$$

Even though the core is hotter than this mean value, temperatures are not high enough, at this stage, to ignite hydrogen fusion.

The opacity is high at this stage (largely due to H^-), as is the luminosity. As a consequence, the system is effectively fully convective, and can therefore be well approximated by a polytrope with $n = 1.5$ (Section 3.2.1).

4.4 Protostars: Hayashi tracks

The relevant behaviour of the protostar in this phase was investigated by the Japanese astronomer Chushiro Hayashi. We investigate this behaviour through a polytropic model.

4.4.1 Interior properties

For a polytrope of index n,

$$P = K\rho^{(n+1)/n}, \tag{3.3}$$

$$= \frac{\rho}{\mu m(\mathrm{H})}kT \tag{1.2}$$

if gas pressure dominates. Eliminating ρ between eqtns. (3.3) and (1.2) we obtain

$$P = K^{-n}\left(\frac{k}{\mu m(\mathrm{H})}\right)^{(1+n)} T^{(1+n)}; \tag{4.9}$$

that is,

$$P = C_1 T^{(1+n)} \tag{4.10}$$

where

$$C_1 = K^{-n}\left(\frac{k}{\mu m(\mathrm{H})}\right)^{(1+n)}$$

is a constant for a given model.

From our mass-radius relation for polytropes,

$$R^{(3-n)} M^{(n-1)} \propto K^n, \tag{3.20}$$

we have, for $n = 1.5$,

$$K \propto M^{1/3} R$$

whence

$$\begin{aligned} C_1 &\propto \left(M^{1/3} R\right)^{-n}, \\ &\propto M^{-1/2} R^{-3/2} \end{aligned}$$

for $n = 1.5$. So, finally, from eqtn. (4.10),

$$P = C_2 M^{-1/2} R^{-3/2} T^{5/2} \tag{4.11}$$

where C_2 is a constant (for given polytropic index and mean molecular weight).

4.4.2 Boundary condition

To solve for the constant of proportionality in eqtn. (4.11), we consider an outer boundary condition – the photosphere. The photosphere is defined by optical depth $\tau = 2/3$; that is, since

$$\tau \equiv \int_r^\infty \kappa \rho \mathrm{d}r,$$

and assuming constant opacity in the atmosphere,

$$\frac{2}{3} = \kappa \int_R^\infty \rho \mathrm{d}r.$$

From hydrostatic equilibrium,

$$\begin{aligned} P(R) &= \int_R^\infty g \rho \mathrm{d}r \\ &\simeq \frac{GM}{R^2} \int_R^\infty \rho \mathrm{d}r \\ &= \frac{GM}{R^2} \frac{2}{3\kappa}. \end{aligned}$$

We've assumed constant κ in the atmosphere of a given protostar, but it will vary between different objects according to pressure and temperature. We adopt a power-law dependence,

$$\kappa = \kappa_0 P^a T^b,$$

as a plausible generic opacity law, so

$$P(R) = \frac{GM}{R^2}\frac{2}{3\kappa_0 P^a T^b}$$

or

$$P(R) = \left(\frac{GM}{R^2}\frac{2}{3\kappa_0}T_{\rm eff}^{-b}\right)^{1/(1+a)} \tag{4.12}$$

(where $T \equiv T_{\rm eff}$ at the photosphere).

4.4.3 Solution

At the photosphere, both eqtn. (4.11) and (4.12) are true; i.e.,

$$\left(\frac{GM}{R^2}\frac{2}{3\kappa_0}T_{\rm eff}^{-b}\right)^{1/(1+a)} = C_2 M^{-1/2} R^{-3/2} T_{\rm eff}^{5/2}.$$

So, for any given mass, there is a single-valued relationship between R and $T_{\rm eff}$; but $L = 4\pi R^2 \sigma T_{\rm eff}^4$, so this is equivalent to a single-valued relationship between $T_{\rm eff}$ and L – that is, a track in the HR diagram. Such tracks are called *Hayashi tracks.* Taking logs on both sides, some algebra yields

$$\ln T_{\rm eff} = A \ln L + B \ln M + \text{constant} \tag{4.13}$$

with

$$A = \frac{0.75a - 0.25}{5.5a + b + 1.5}, \qquad B = \frac{0.5a + 1.5}{5.5a + b + 1.5}.$$

Opacity calculations indicate $a \simeq 1$, $b \simeq 3$ (the principal source of opacity is H^-) whence
$A \simeq 0.05$
$B \simeq 0.2$
From eqtn. 4.13,

$$\frac{\partial \ln L}{\partial \ln T_{\rm eff}} = 1/A;$$

since A is small, the track must be steep (i.e., nearly constant temperature) in the HR diagram. We also see that

$$B = \frac{\partial \ln T_{\rm eff}}{\partial \ln M}$$

is positive, so that the tracks move slightly to the left with increasing mass; but the dependence is weak, so all fully-convective stars lie on almost the same 'Hayashi track' (or Hayashi line). For a given mass and chemical composition, no fully convective star can lie to the right of the Hayashi track (because convection is the most efficient means of energy transport). The region to the right of the Hayashi track is the Hayashi 'forbidden zone' ($T_{\rm eff} \lesssim 4$ kK).

4.5 Henyey track

As the contracting protostar descends the Hayashi track the internal temperature continues to rise until ionization is complete, and the opacity drops in the core. This fall in opacity allows energy to be transported radiatively for solar-type stars, and the nascent star develops a radiative core. The star then moves away from the Hayashi track, to higher $T_{\rm eff}$, following a near-horizontal 'Henyey' track (for $M \gtrsim 0.5{\rm M}_\odot$).

From the equation of radiative energy transport

$$L(r) \propto \frac{r^2}{\overline{\kappa}_{\rm R}\rho}\frac{{\rm d}T}{{\rm d}r}T^3, \tag{2.9}$$

but we also have

$$\rho \propto \frac{M}{R^3},$$
$$\frac{{\rm d}T}{{\rm d}r} \simeq \frac{T_{\rm c}}{R},$$

and we adopt

$$\overline{\kappa}_{\rm R} \propto \rho T^{-3.5}$$

(Kramer's opacity). Finally, the core temperature scales as

$$T_{\rm c} \propto \frac{M}{R} \tag{20.17}$$

(from PHAS2112). Combining these results yields

$$L \propto M^{5.5} R^{-0.5}$$

and (since $L \propto R^2 T_{\rm eff}^4$)

$$T_{\rm eff} \propto R^{-5/8}. \tag{4.14}$$

That is, for given mass M, the luminosity and temperature increase as the star shrinks.

4.6 Protostar to star

Until fusion ignites, the relevant timescale is the Kelvin-Helmholtz timescale (since the star is radiating gravitational potential energy). This timescale is short, and the contracting protostar is still shrouded in the dusty molecular cloud from which it formed. The increasing core temperature eventually results in ignition of hydrogen burning; the protostar is now a star, on the zero-age main sequence (ZAMS).

Of course, the true circumstances are more complex in detail than the simple Hayashi picture; protostars show circumstellar accretion disks, and outflows such as jets and stellar winds. Magnetic fields also play a role. Material falling onto the protostar generates accretion luminosity, potentially through shocks. This extra energy loss results in protostellar radii smaller than simple estimates (eqtn. 4.8). Furthermore, the most massive stars may ignite hydrogen burning at the core while still accreting at the surface.

Stars more massive than $\sim 5\mathrm{M}_\odot$ become stable against convection very quickly; most of their pre-main-sequence evolution is on the Henyey branch, while stars with $M \lesssim 0.5\mathrm{M}_\odot$ never become stable against convection, and evolve vertically onto the MS as fully convective stars.

comes a larger fraction of the total pressure and electron scattering replaces free-free and bound-free absorption as the main contributor to interior opacity; the adiabatic temperature gradient in the convective core and the radiative gradient outside of this core approach one another more closely so that the changes in structure accompanying variations in the core mass become less radical.

As in the case of $M = 1.5\ M_{\odot}$, the point at which luminosity attains its final minimum, before increasing during the main hydrogen-burning stages, may be defined as the

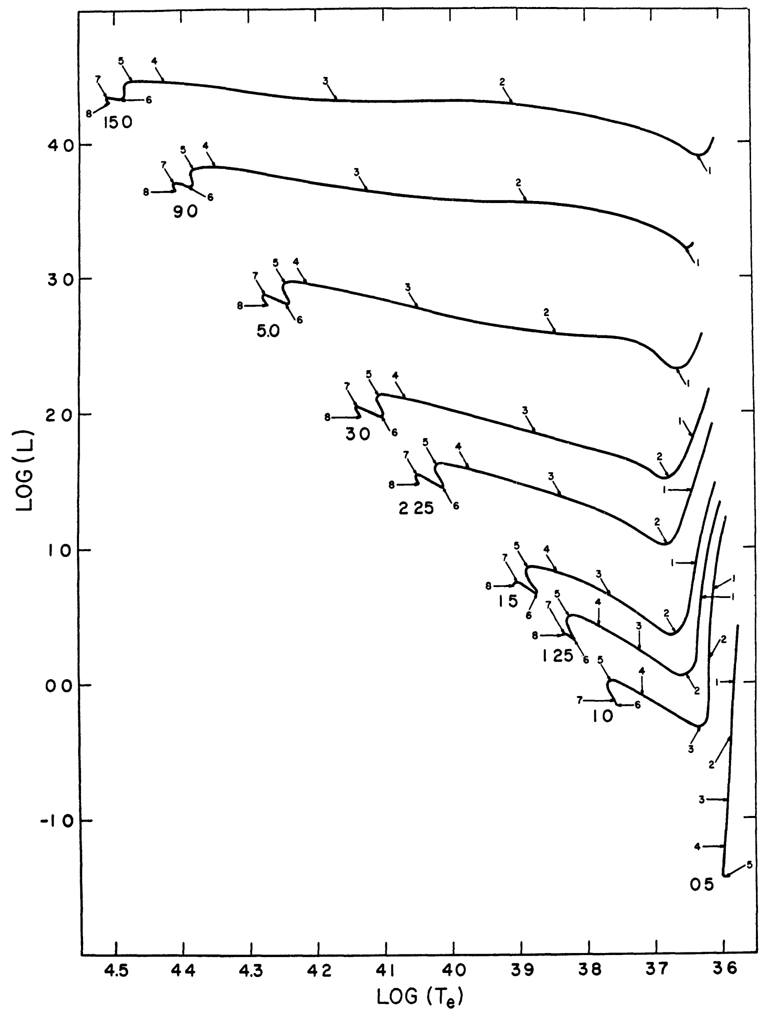


FIG. 17.—Paths in the Hertzsprung-Russell diagram for models of mass $(M/M_{\odot})$ = 0.5, 1.0, 1.25, 1.5, 2.25, 3 0, 5.0, 9.0, and 15.0. Units of luminosity and surface temperature are the same as those in Fig. 1

Figure 4.1: The approach to the main sequence for (proto)stars of different masses. From Iben, ApJ, 141, 993 (1965).

Section 5

The main sequence: homologous models

We know that stars on the main sequence share many basic characteristics, as a consequence of their common energy source. We also know that many properties vary along the main sequence, i.e., vary with mass. Both aspects are accommodate by supposing that stars along the main sequence are essentially just scaled version of each other. This will give us insight into (e.g.) main-sequence mass–luminosity relationships.

5.1 Transformed equations

The basic equations of stellar structure are normally cast in such a way as to describe the run of physical properties with radius; but *mass* is the more fundamental physical property of a star (the radius of a solar-mass star will change by orders of magnitude over its lifetime, while its mass remains more or less constant), so for practical purposes it is customary to reformulate these structure equations in terms of mass as the independent variable. We start by simply inverting eqtn. 1.4, the equation of mass continuity, giving

$$\frac{\mathrm{d}r}{\mathrm{d}m(r)} = \frac{1}{4\pi r^2 \rho(r)}; \tag{5.1}$$

and the remaining basic structure equations are just multiplied by eqtn. 5.1, giving

$$\frac{\mathrm{d}P(r)}{\mathrm{d}m(r)} = \frac{-Gm(r)}{4\pi r^4} \qquad \text{(hydrostatic equilibrium)} \tag{5.2}$$

$$\frac{\mathrm{d}L(r)}{\mathrm{d}m(r)} = \varepsilon(r) \qquad \text{(energy continuity)} \tag{5.3}$$

$$\frac{\mathrm{d}T(r)}{\mathrm{d}m(r)} = -\frac{3\overline{k}_{\mathrm{R}} L(r)}{16\pi^2 r^4 acT^3(r)} \qquad \text{(radiative energy transport)} \tag{5.4}$$

(where all the radial dependences have been shown explicitly).

5.2 Homologous models

Homologous stellar models are defined such that their properties scale in the same way with fractional mass $\mathtt{m} \equiv m(r)/M$. That is, for some property X (which might be temperature, or density, etc.), a plot of X vs. $\mathtt{m}$ is the same for all homologous models. Such models may reasonably be applied to zero-age main-sequence stars which have uniform chemical composition and which are in thermal, hydrostatic, and radiative equilibrium.

Our aim in constructing such models is to formulate the stellar-structure equations so that they are independent of absolute mass, but depend only on *relative* mass. We therefore recast the variables of interest as functions of fractional mass, with the dependences on *absolute* mass assumed to be power laws:

$$\begin{aligned} r &= M^{x_1} r_0(\mathtt{m}) & \mathrm{d}r &= M^{x_1}\mathrm{d}r_0 \\ \rho(r) &= M^{x_2}\,\rho_0(\mathtt{m}) & \mathrm{d}\rho(r) &= M^{x_2}\mathrm{d}\rho_0 \\ T(r) &= M^{x_3}\,T_0(\mathtt{m}) & \mathrm{d}T(r) &= M^{x_3}\mathrm{d}T_0 \\ P(r) &= M^{x_4}\,P_0(\mathtt{m}) & \mathrm{d}P(r) &= M^{x_4}\mathrm{d}P_0 \\ L(r) &= M^{x_5}\,L_0(\mathtt{m}) & \mathrm{d}L(r) &= M^{x_5}\mathrm{d}L_0 \end{aligned} \tag{5.5}$$

where the x_i exponents are constants to be determined, and r_0, ρ_0 etc. depend only on the fractional mass $\mathtt{m}$. We also have

$$m(r) = \mathtt{m}M \qquad \mathrm{d}m(r) = M\mathrm{d}\mathtt{m}$$

and, from eqtns. (3.1, 3.2)

$$\begin{aligned} \kappa(r) &= \kappa_0\rho^a(r)T^b(r) & &= \kappa_0\rho_0^a(\mathtt{m})T_0^b(\mathtt{m})M^{ax_2}M^{bx_3} \\ \varepsilon(r) &\simeq \varepsilon_0\rho(r)T^\alpha(r), & &= \varepsilon_0\rho_0(\mathtt{m})T_0^\alpha(\mathtt{m})M^{x_2}M^{\alpha x_3}, \end{aligned}$$

We can now substitute these into our structure equations to express them in terms of dimensionless mass $\mathtt{m}$ in place of actual mass $m(r)$:

<u>*Mass continuity:*</u>

$$\frac{\mathrm{d}r}{\mathrm{d}m(r)} = \frac{1}{4\pi r^2\rho(r)} \qquad \text{becomes} \tag{5.1}$$

$$M^{(x_1-1)}\frac{\mathrm{d}r_0(\mathtt{m})}{\mathrm{d}\mathtt{m}} = \frac{1}{4\pi r_0^2(\mathtt{m})\rho_0(\mathtt{m})}M^{-(2x_1+x_2)} \tag{5.6}$$

The condition of homology requires that the scaling be independent of actual mass, so we can equate the exponents of M on either side of the equation to find

$$3x_1 + x_2 = 1 \tag{5.7}$$

Hydrostatic equilibrium:

$$\frac{\mathrm{d}P(r)}{\mathrm{d}m(r)} = \frac{-Gm(r)}{4\pi r^4} \qquad \text{becomes} \tag{5.2}$$

$$M^{(x_4-1)}\frac{\mathrm{d}P_0(\mathtt{m})}{\mathrm{d}\mathtt{m}} = -\frac{G\mathtt{m}}{4\pi r_0^4}M^{(1-4x_1)} \tag{5.8}$$

$$\text{whence } 4x_1 + x_4 = 2 \tag{5.9}$$

Energy continuity:

$$\frac{\mathrm{d}L(r)}{\mathrm{d}m(r)} = \varepsilon(r)$$

$$= \varepsilon_0\rho(r)T^\alpha(r) \qquad \text{becomes} \tag{5.3}$$

$$M^{(x_5-1)}\frac{\mathrm{d}L_0(\mathtt{m})}{\mathrm{d}\mathtt{m}} = \varepsilon_0\rho_0(\mathtt{m})T_0^\alpha(\mathtt{m})M^{x_2+\alpha x_3} \tag{5.10}$$

$$\text{whence } x_2 + \alpha x_3 + 1 = x_5 \tag{5.11}$$

Radiative transport:

$$\frac{\mathrm{d}T(r)}{\mathrm{d}m(r)} = -\frac{3\overline{k}_\mathrm{R}L(r)}{16\pi^2r^4acT^3(r)} \qquad \text{becomes} \tag{5.4}$$

$$M^{(x_3-1)}\frac{\mathrm{d}T_0(\mathtt{m})}{\mathrm{d}\mathtt{m}} = -\frac{3(\kappa_0\rho_0^aT_0^b)L_0}{16\pi^2r_0^4acT_0^3(\mathtt{m})}M^{(x_5+(b-3)x_3+ax_2-4x_1)} \tag{5.12}$$

$$\text{whence } 4x_1 + (4-b)x_3 = ax_2 + x_5 + 1 \tag{5.13}$$

Equation of state:

If gas pressure dominates, then

$$P_\mathrm{G}(r) = n(r)kT(r)$$

$$= \frac{\rho(r)kT(r)}{\mu m(\mathrm{H})}; \tag{1.2}$$

neglecting any radial dependence of μ we find

$$M^{x_4}P_0(\mathtt{m}) = \frac{\rho_0(\mathtt{m})kT_0(\mathtt{m})}{\mu m(\mathrm{H})}M^{(x_2+x_3)} \tag{5.14}$$

$$\text{whence } x_2 + x_3 = x_4. \tag{5.15}$$

Alternatively, if radiation pressure dominates,

$$P_\mathrm{R}(r) = \frac{1}{3}aT^4(r) \tag{1.3}$$

$$M^{x_4}P_0(\mathtt{m}) = \frac{1}{3}a\left(M^{x_3}T_0(\mathtt{m})\right)^4 \tag{5.16}$$

$$\text{whence } x_4 = 4x_3 \tag{5.17}$$

5.3 Solutions

We now have five equations[1] for the five exponents x_i, which can be solved for given values of a, b, and α (that is, the exponents in eqtns. (3.1) and (3.2)).

For low-mass stars ($\sim 0.7 \lesssim M \lesssim 2\mathrm{M}_\odot$, corresponding roughly to spectral types F and later)[2] we adopt Kramers opacity

$$\kappa \propto \rho T^{-3.5}$$

(i.e., $a = 1$, $b = -3.5$) and a nuclear generation rate appropriate to proton-proton fusion,

$$\epsilon \propto T^4$$

($\alpha = 4$).

Higher-mass stars have energy generation on the main sequence dominated by the CNO cycle, where

$$\epsilon \propto T^{16},$$

and we suppose that the opacity is dominated by simple Thompson scattering (mass opacity independent of density and temperature; $a = b = 0$).[3]

The algebra to obtain solutions is straightforward (if, in general, tedious). Results for the cases of 'low-mass' and 'high-mass' stars are:

Regime	a	b	α	x_1	x_2	x_3	x_4	x_5
Low-mass	1	-3.5	4	1/13	10/13	12/13	22/13	71/13
High-mass	0	0	16	15/19	$-26/19$	4/19	$-22/19$	3

[1] Eqtns. (5.7), (5.9), (5.11), (5.13), and (5.15/5.17)

[2] For masses much below $0.7\mathrm{M}_\odot$ energy transport is convective, so our numerical analysis, which uses radiative transport, becomes invalid.

[3] Our analysis breaks down again for very high masses, when radiation becomes the dominant source of pressure.

5.4 Results

Collecting the set structure equations that describe a sequence of homologous models:

$$\frac{\mathrm{d}r_0(\mathtt{m})}{\mathrm{d}\mathtt{m}} = \frac{1}{4\pi r_0^2(\mathtt{m})\rho_0(\mathtt{m})} \tag{5.6}$$

$$\frac{\mathrm{d}P_0(\mathtt{m})}{\mathrm{d}\mathtt{m}} = -\frac{G\mathtt{m}}{4\pi r_0^4} \tag{5.8}$$

$$\frac{\mathrm{d}L_0(\mathtt{m})}{\mathrm{d}\mathtt{m}} = \varepsilon_0\rho_0(\mathtt{m})T_0^\alpha(\mathtt{m}) \tag{5.10}$$

$$\frac{\mathrm{d}T_0(\mathtt{m})}{\mathrm{d}\mathtt{m}} = -\frac{3(\kappa_0\rho_0^a T_0^b)L_0}{16\pi^2 r_0^4 acT_0^3(\mathtt{m})} \tag{5.12}$$

$$P_0(\mathtt{m}) = \frac{\rho_0(\mathtt{m})kT_0(\mathtt{m})}{\mu m(\mathrm{H})}. \tag{5.14}$$

These can be solved numerically using the boundary conditions

$$r_0 = 0, \qquad L_0 = 0 \text{ at } \mathtt{m} = 0,$$
$$P_0 = 0, \qquad \rho_0 = 0 \text{ at } \mathtt{m} = 1.$$

However, we can draw some useful conclusions analytically. First, it's implicit in our definition of homologous models, eqtns (5.5), that there must exist mass–luminosity and mass-radius relations for them; since

$$L = M^{x_5}L_0(1) \qquad\qquad R = M^{x_1}r_0(1)$$

it follows immediately that

$$L \propto M^{71/13} \sim M^{5.5} \qquad\qquad R \propto M^{1/13} \sim M^{0.1} \text{ (low-mass)}$$
$$\propto M^3 \qquad\qquad R \propto M^{15/19} \sim M^{0.8} \text{ (high-mass)}$$

which is not too bad compared to the actual main-sequence mass–luminosity relationship.

Secondly, since

$$L \propto R^2 T_{\mathrm{eff}}^4 \quad \text{and} \qquad R = M^{x_1}r_0(1),$$
$$L = M^{x_5}L_0(1),$$

it follows that

$$M^{x_5}L_0(1) \propto M^{2x_1}r_0^2(1)T_{\mathrm{eff}}^4,$$

or

$$T_{\rm eff} \propto M^{(x_5-2x_1)/4}, \qquad \sim \propto M^{1.2} \text{ (low-mass)}$$
$$\sim \propto M^{0.35} \text{ (high-mass)}$$

Combining this with the mass–luminosity relationship gives

$$L \propto T_{\rm eff}^{4x_5/(x_5-2x_1)} \qquad \sim \propto T_{\rm eff}^{4.5} \text{ (low-mass)}$$
$$\sim \propto T_{\rm eff}^{8.5} \text{ (high-mass)}$$

which is not too bad – indeed, the qualitative result that there *is* a luminosity–temperature relationship is, in effect, a prediction that a main sequence exists in the HR diagram.[4]

[4]Although there is some circularity in this argument, since we supposed homology as a starting premise.

Section 6

Stellar evolution

6.1 Mass limits for stars

We showed in Section 3.4 that the central density and pressure of a star are readily estimated for a polytropic model, with

$$\overline{\rho} = \frac{3M}{4\pi R^3} = \rho_c \frac{3}{\xi_1^3} \left\{ -\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\bigg|_{\xi=\xi_1} \right\}; \tag{3.18}$$

i.e.,

$$\rho_c = \frac{\xi_1^3 M}{4\pi R^3} \left\{ -\xi^2 \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\bigg|_{\xi=\xi_1} \right\}^{-1}$$

$$P_c = K\rho_c^{(n+1)/n}, \tag{3.3}$$

$$= \left\{ 4\pi(n+1) \left(\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\bigg|_{\xi_1} \right)^2 \right\}^{-1} \frac{GM^2}{R^4}. \tag{3.19}$$

For a given polytropic index n we can therefore estimate the core temperature (e.g., through the perfect gas equation if gas pressure dominates). These estimates of core conditions allow us to estimate the minimum mass for which thermonuclear fusion can take place ($T_c \simeq 10^6$ K); this turns out to be $\sim$0.1M$_\odot$.

There's also an upper limit to the possible mass of a star. Classically, this is determined by the Eddington limit. The inward force of gravity is proportional to mass, while the outward radiation-pressure force scales with luminosity, which increase with mass to the power $\sim$3; thus there must be a point at which radiation overcomes gravity, and the star cannot be stable. This limit is generally thought to be at $\sim$150M$_\odot$. Only a few dozen stars are known that are more

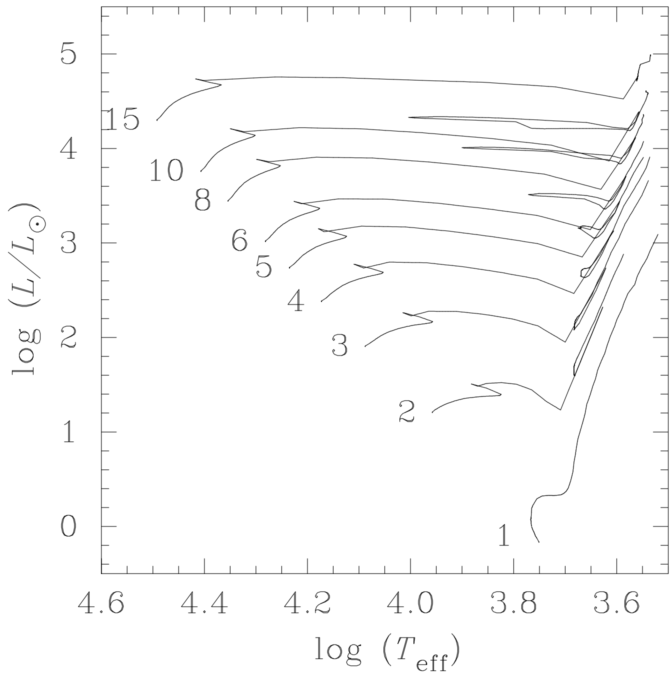


Figure 6.1: Evolutionary tracks for stars of different initial masses, starting at the ZAMS (based on calculations by the Geneva group). The tracks of intermediate-mass stars, with initial masses of $\sim$2–10M$_\odot$, all show broadly similar characteristics.

massive than 100M$_\odot$, although *the* most massive star has a reported mass of $\sim$200M$_\odot$. Possibly this mass estimate is in error, or perhaps there are ways round the Eddington limit (e.g., inhomogeneous atmospheres that allow radiation to 'leak' out through lower-density routes).

6.2 Classifying stellar evolution

Within this allowed range of $\sim 10^{-1}$–10^2M$_\odot$, a star, by our definition, spends most of its lifetime on the main sequence, burning hydrogen to helium in the core. However, even on the main sequence there are different evolutionary behaviours, depending on mass. In this context, it's convenient to divide the MS into the 'upper' main sequence ($M \gtrsim 2$M$_\odot, T_{\rm eff} \gtrsim 10^4$ K) and 'lower' main sequence:

> *Lower main sequence.* Spectral types $\sim$F and later. Energy generation is through the proton-proton chain, with radiative cores and convective envelopes. The extent of the envelope varies with mass; at 1M$_\odot$ the envelope accounts for $\sim$3% by mass ($\sim$30% by radius), increasing to $\sim$40% at 0.5M$_\odot$. Stars less massive than $\sim$0.25M$_\odot$are fully convective.

Upper main sequence. Energy generation is through the CNO cycle; the high energy fluxes result in convective cores, but lower envelope opacities result in radiative envelopes. The fractional core mass increases with total mass, because of the strong dependence of energy generation rate on temperature (i.e., mass); $\sim$0.17M at 3$M_\odot$, rising to $\sim$0.38M at 15$M_\odot$.

6.2.1 Mass dependence of subsequent evolution

The discussion of subsequent evolution can be structured in several ways – for example, according to the minimum mass required for a star to form its first degenerate core. We can therefore elaborate our simple 'high-mass/loss-mass' division to discuss this subsequent evolution.

First degenerate cores as a function of mass

Mass range ($M_\odot$)	Category	First degenerate core
$\leq$2	Low	He
2–8	Intermediate	C/O
8–11	High	O/Ne/Mg
$\geq$11	High	–

- Low-mass stars develop a degenerate helium core while ascending the red-giant branch (discussed below; Section 6.4). Core contraction is slow, leading to a slow ascent of the RGB. Core helium ignition takes place explosively in a thermonuclear runway; in this 'helium flash' enormous energy ($\sim 10^{11} L_\odot$!) is generated for a few seconds, lifting the degeneracy; the core expands and helium burning becomes stable. This helium flash isn't directly observable because the energy is absorbed by the envelope, and is released slowly (on a thermal timescale).

  Helium burns to a C/O core, which never gets hot enough to ignite. The star ends its life as a C/O white dwarf.

- Intermediate-mass stars are discussed in detail below. They have core temperatures high enough to ignite helium without going through a degenerate phase. The star again ends as a white dwarf of degenerate C/O.

- High-mass stars are hot enough to ignite core carbon burning before developing O/Ne/Mg degenerate cores. For initial masses in excess of $\sim$11$M_\odot$ subsequent stages of nuclear fusion can occur, all the way to Fe; these stars end their lives as core-collapse supernovae.

6.3 Evolution on the main sequence

Homology implies for two stars 1, 2

$$\frac{r_1(\mathtt{m})}{R_1} = \frac{r_2(\mathtt{m})}{R_2}, \tag{6.1}$$

$$\frac{m_1(\mathtt{m})}{M_1} = \frac{m_2(\mathtt{m})}{M_2} \tag{6.2}$$

whence we can obtain the density scaling from

$$\begin{aligned}\frac{\mathrm{d}m_2}{\mathrm{d}r_2} &= \frac{\mathrm{d}m_1}{\mathrm{d}r_2}\frac{M_2}{M_1} && \text{(from eqtn. 6.2)}\\ &= \frac{\mathrm{d}m_1}{\mathrm{d}r_1}\frac{R_1}{R_2}\frac{M_2}{M_1} && \text{(from eqtn. 6.1).}\end{aligned} \tag{6.3}$$

We also have

$$\begin{aligned}\frac{\mathrm{d}m_2}{\mathrm{d}r_2} &= 4\pi r_2^2\rho_2 && \text{(from mass continuity, eqtn. 1.4)}\\ &= 4\pi r_1^2\left(\frac{R_2}{R_1}\right)^2\rho_2 && \text{(from eqtn. 6.1),}\end{aligned} \tag{6.4}$$

so, from eqtns 6.3 and 6.4,

$$\begin{aligned}\frac{\mathrm{d}m_1}{\mathrm{d}r_1} &= 4\pi r_1^2\left(\frac{M_1}{M_2}\right)\left(\frac{R_2}{R_1}\right)^3\rho_2,\\ &= 4\pi r_1^2\rho_1\end{aligned}$$

or

$$\frac{\rho_2}{\rho_1} = \rho_1\left(\frac{M_2}{M_1}\right)\left(\frac{R_1}{R_2}\right)^3 \tag{6.5}$$

(where 'ρ_1' = $\rho_1(\mathtt{m})$', etc.). Mass continuity and hydrostatic equilibrium give

$$\frac{\mathrm{d}P}{\mathrm{d}m} = \frac{-Gm}{4\pi r^4} \tag{5.2}$$

$$\tag{6.6}$$

so

$$\begin{aligned}\frac{\mathrm{d}P_1}{\mathrm{d}m_1} &\equiv \frac{\mathrm{d}P_2}{\mathrm{d}m_2}\frac{\mathrm{d}m_2}{\mathrm{d}m_1}\frac{\mathrm{d}P_1}{\mathrm{d}P_2}\\ &= \frac{-Gm_2}{4\pi r_2^4}\left(\frac{M_2}{M_1}\right)\frac{\mathrm{d}P_1}{\mathrm{d}P_2}\\ &= \frac{-Gm_1}{4\pi r_1^4}; && \text{that is,}\\ \frac{\mathrm{d}P_2}{\mathrm{d}P_1} &= \left(\frac{M_2}{M_1}\right)\left(\frac{m_2}{m_1}\right)\left(\frac{r_1}{r_2}\right)^4\\ &= \left(\frac{M_2}{M_1}\right)^2\left(\frac{R_1}{R_2}\right)^4.\end{aligned}$$

Integrating (and taking the surface pressure to be zero),

$$\frac{P_2}{P_1} = \left(\frac{M_2}{M_1}\right)^2 \left(\frac{R_1}{R_2}\right)^4. \tag{6.7}$$

Finally, for an equation of state

$$P_1 = (\rho_1 k T_1)/(\mu_1 m(\mathrm{H})) \tag{1.2}$$

we find, from eqtns. (6.5) and (6.7),

$$\frac{T_2}{T_1} = \frac{\mu_2}{\mu_1}\frac{M_2}{M_1}\frac{R_1}{R_2} \tag{6.8}$$

(provided that μ_1/μ_2 is independent of radius).

Evolution on the main sequence is driven by the slow (nuclear-timescale) conversion of hydrogen to helium in the core, with a consequent increase in mean molecular weight. To maintain the pressure (i.e., to maintain hydrostatic equilibrium) the core contracts, causing the core temperature to rise (eqtn. 6.8).

Because of the temperature dependence of energy generation, this results in an increase in luminosity, and the star moves upwards in the HRD. (Now half-way through its main-sequence lifetime, the Sun is currently about 30% more luminous than it was on the zero-ag main sequence.) The shrinking core is also accompanied by expansion of the envelope (the 'mirror effect'; Section 6.4.1).

Precise details of main-sequence evolution depend on the effectiveness of mixing in the core. This can be seen in the different evolutionary tracks shown in Fig. 6.1; for lower main-sequence stars ($M \lesssim 2\mathrm{M}_\odot$), there is little or no mixing in the radiative core, and the mean molecular weight in the centre builds up relatively quickly. Energy production becomes concentrated in a 'thick shell' around the centre. Consequently, there is a gradual transition to subsequent evolutionary phases where fusion occurs in a shell around a core consisting of helium (and heavier elements).

More massive stars have fully mixed (convective) cores, so exhaust hydrogen at roughly the same time throughout the core. In the final stages of core hydrogen fusion the entire star contracts to try to maintain energy generation, increasing T_{eff} and producing a short-lived 'blue hook' in the HRD. Again, as core hydrogen is exhausted, the star establishes a hydrogen-burning shell around the core.

The main-sequence lifetime is

$$\tau \propto M/L \propto M^{(1-x_5)}$$

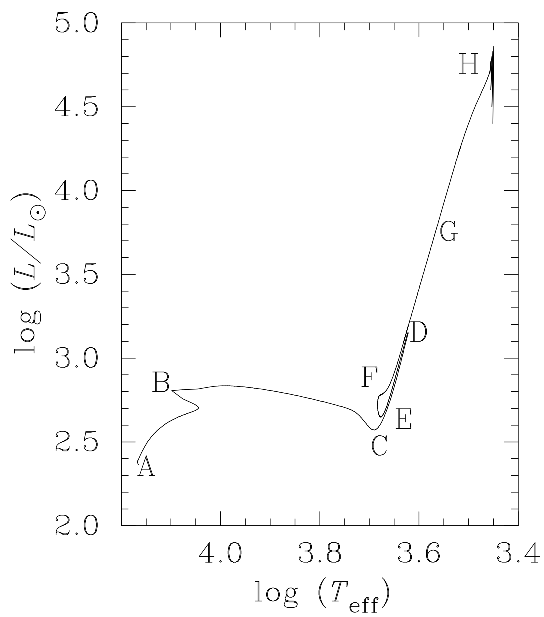


Figure 6.2: Evolutionary track for a 4-$M_\odot$ star (calculated using `EZ-Web`, schematically extended to include the thermal-pulsing AGB at `H`).

for our homologous models (i.e., $\tau \propto M^{-2}$ for high mass stars); more massive star have shorter lifetimes than less mass ones. The main-sequence lifetime of the Sun is $\sim 10^{10}$ yr, while a massive star is much less ($\sim 10^6$–10^7 yr for $\sim 20M_\odot$).

The evolution of the most massive (O- and B-type) stars are also affected by mass loss through stellar winds. A $25M_\odot$ star with a main-sequence lifetime of 6×10^6 yr losing mass at a rate of $\sim 10^{-6}\ M_\odot\ \mathrm{yr}^{-1}$ will lose a quarter of its ZAMS mass in this way. The paradoxical consequence of losing mass is that the star's main-sequence lifetime is extended.

6.4 Evolution off the main sequence: intermediate-mass stars

6.4.1 Red-giant branch: shell hydrogen burning

As an example of post-main-sequence evolution, we'll discuss in detail intermediate-mass stars, which illustrate a number of features of interest (specifically, a $4M_\odot$ star; Fig. 6.2). When core hydrogen is exhausted (the 'terminal age main sequence', or TAMS; point `B` in Fig. 6.2) the star has, essentially, an inert helium core surrounded by a hydrogen-burning shell. The core mass increases (gaining mass from the shell), and contracts under gravity. As the core contracts, the response of the envelope is to expand: the 'mirror principle' (or 'shell-burning law').

Evolutionary timescales for a 4-$M_\odot$ star

(based on Geneva models)

Phase	Fig. 6.2	t (yr)	Δt	Energy source
Main sequence	A–hook	1.62+8	–	Core H burning
	hook–B	1.65+8	5.30+6	
Hertzsprung gap	B–C	1.66+8	0.97+6	Shell H burning
RGB	C–D	1.66+8	0.66+6	
CHeB	D–E	1.74+8	8.11+6	Core He + shell H
CHeB	E–F	1.92+8	1.77+7	
AGB	F–G	1.94+8	1.78+6	Shell He + pulsing

All numerical stellar-evolution models predict this transition, and yet we lack a *simple,* didactic physical explanation:

> "Why do some stars evolve into red giants though some do not? This is a classic question that we consider to have been answered only unsatisfactorily." – *D. Sugimoto & M. Fujimoto (ApJ, 538, 837, 2000)*

Nevertheless, semi-phenomenological descriptions afford some insight; these can be presented in various degrees of detail, of which the most straighforward argument is as follows.

We simplify the stellar structure into an inner core and an outer envelope, with masses and radii $M_\mathrm{c}, M_\mathrm{e}$ and $R_\mathrm{c}, R_\mathrm{e} (= R_*)$ respectively. At the end of core hydrogen burning, we suppose that core contraction happens quickly – faster than the Kelvin-Helmholtz timescale, so that the virial theorem holds, and thermal and gravitational potential energy are conserved to a satisfactory degree of approximation. We formalize this supposition by writing

$$\Omega + 2U = \text{constant} \qquad \text{(Virial theorem)}$$
$$\Omega + U = \text{constant} \qquad \text{(Energy conservation)}$$

These two equalities can only hold simultaneously if both Ω and U are individually constant, summed over the whole star. In particular, the total gravitational potential energy is constant

Stars are centrally condensed, so we make the approximation that $M_\mathrm{c} \gg M_\mathrm{e}$; then, adding the core and envelope,

$$|\Omega| \simeq \frac{GM_\mathrm{c}^2}{R_\mathrm{c}} + \frac{GM_\mathrm{c}M_\mathrm{e}}{R_*}$$

We are interested in evolution, so we take the derivative with respect to time,

$$\frac{\mathrm{d}|\Omega|}{\mathrm{d}t} = 0 = -\frac{GM_\mathrm{c}^2}{R_\mathrm{c}^2}\frac{\mathrm{d}R_\mathrm{c}}{\mathrm{d}t} - \frac{GM_\mathrm{c}M_\mathrm{e}}{R_*^2}\frac{\mathrm{d}R_*}{\mathrm{d}t}$$

i.e.,

$$\frac{\mathrm{d}R_*}{\mathrm{d}R_\mathrm{c}} = -\frac{M_\mathrm{c}}{M_\mathrm{e}}\left(\frac{R_*}{R_\mathrm{c}}\right)^2.$$

The negative sign demonstrates that as the core *contracts*, the envelope must *expand* – a good rule of thumb throughout stellar evolution, and, in particular, what happens at the end of the main sequence for solar-type stars.

As the radius increases, the effective temperature falls, and the star moves rightwards across the HRD. The temperature and density gradients between core and envelope are initially shallow, and the shell is quite extensive (in mass); this phase is referred to as 'thick-shell burning'. As the core contracts and the envelope expands, these gradients increase, and the shell occupies less mass. During this 'thin-shell burning', a significant part of the energy goes into expanding the envelope, leading to a drop in luminosity.

The increasing opacity that accompanies cooling temperatures favours convection, and the star approaches the Hayashi track (Fig. 6.2, point `C`). As the core continues to shrink, and the envelope expands in response, further decreases in T_eff are not possible (the star can't transport energy efficiently enough), so the luminosity increases and the star ascends the 'red-giant branch' (RGB).

The transition from main sequence to red giant is rapid, so few stars are observed in this region of the HRD (points `B-C`) – the so-called 'Hertzsprung gap'. The expansion `C-D` occurs on a thermal timescale, so the hydrogen shell-burning phase is short-lived for intermediate-mass stars (it is much longer for lower-mass stars).

First dredge-up

As an intermediate-mass star ascends the RGB, it develops an extensive convective envelope which reaches down to the hydrogen-burning shell (echoing the fully convective phase of the Hayashi track), bring CNO-processed material to the surface - the *first dredge-up*. Because the C–N cycle reaches equilibrium before the O–N cycle, C–N processed material is exposed at the surface; surface N is typically enriched by a factor ∼2, C is depleted by ∼30% (and O is unchanged). The observation of this CN-processed material is important evidence that it is the CNO cycle (not proton-proton burning) that occurs in the H-burning shell.

Stellar winds

Red giants are observed to lose mass in the form of slow winds ($v \simeq 5$–30 km s^{-1}, $\dot{M} \sim 10^{-8}$ M$_\odot$ yr^{-1}). Depending on luminosity, the star can lose several tenths of a solar mass through this wind.

6.4.2 Horizontal branch: core helium (+shell hydrogen) burning

The helium 'ashes' of shell hydrogen burning increase the mass of the core, which contracts under increasing pressure. According to the virial theorem, half the energy released by the gravitational contraction of the core is radiated away, and half goes into heating the core. Eventually the core temperature is high enough to ignite core helium burning in the triple-α process ($T \sim 10^8$ K; point `D`).[1] The star now has two energy sources: the helium-burning core, and the hydrogen-burning shell.

The core expands as a result of the energy input (heating) and the density in the adjacent hydrogen shell drops. The shell contributes most ($\sim$70%) of the energy, so the drop in density leads to a fall in total energy generation (luminosity), while the shrinkage of the envelope (mirror law) leads to to a rise in effective temperature – the star moves down and to the left in the HRD (to point `E`).

The star is now in a relatively stable stage of core-helium (+shell-hydrogen) burning (`E-F`). In the HRDs of globular clusters (coeval low-metallicity systems), these stars form a 'horizon branch', as a result of bluewards loops at constant luminosity (not obvious for the 4-$M_\odot$ track shown in Fig. 6.2, but clear for the 6–10-$M_\odot$ tracks in Fig. 6.1); this relatively long-lived phase ($\sim$10% of the main-sequence lifetime) is therefore called the *horizontal branch*.[2]

6.4.3 Early asymptotic giant branch (E-AGB): shell helium burning `[G]`

Eventually core helium is exhausted (point `F`), leaving an inert core composed principally of carbon and oxygen. Helium continues to burn outside the core, in a thick shell. Core contraction is accompanied by expansion of the envelope, and the drop in density and temperature quenches the shell hydrogen burning; the sole energy source is now the helium-burning shell. Cooling of the envelope is accompanied by an increase in opacity, and the envelope becomes strongly convective; the star moves back onto the Hayashi track, and ascends the *asymptotic giant branch* (AGB; `G`). For intermediate-mass stars, the contracting CO core becomes degenerate (Table 6.2.1). The AGB phase lasts about 10% of the horizontal-branch lifetime.

---

[1] For low-mass stars $M \sim$0.8–2.3$M_\odot$) the core is degenerate, with a mass of $\sim$0.45$M_\odot$; the degeneracy is lifted by the ignition, in a 'helium flash', which raises the core luminosity to as much as 10^{10}–$10^{11} L_\odot$, but only briefly (a few minutes).

[2] The horizontal branch of the globular cluster 47 Tuc (Fig. 1.1) is evident at $V \simeq 13.8$, $(B-V) \simeq 0.7$. The stars at the red end of the horizontal branch (typically of higher metallicity than globular-cluster stars) form a 'red clump', used as a distance indicator for extragalactic systems in particular.

Paczynski relation.

The star's luminosity at this stage is largely determined not by the total mass, but by the mass of the degenerate core (although the latter is a function of the former, of course). We know that there exists a mass–radius relationship for degenerate bodies (Section 3.4.1); thus the pressure, and hence the energy-generation rate, in a shell outside the core is principally dependent on the core mass. Paczynski (1970) quantified this idea; a modern version (from Langer) is

$$L_* \simeq 5.9 \times 10^4 \, (M_c/M_\odot - 0.52) \, L_\odot$$

(where M_c is the core mass).

Stars with higher initial masses will have higher-mass cores, and hence higher AGB luminosities. Since, ultimately, the luminosity of the helium-burning shell drives the envelope expansion, stars with higher core masses also evolve to larger radii ($\sim$300$R_\odot$ for the model shown in Fig. 6.2).

Second dredge-up.

The convective envelope reaches down to the now-dormant hydrogen-burning shell, cycling the CNO-processed material to the surface. In this *second dredge-up*, surface abundances of He and N are enhanced, and C and O depleted. (The second dredge-up only occurs for stars with initial masses $\gtrsim$3–4$M_\odot$; at lower masses, the convective zone doesn't extend down to the hydrogen discontinuity.)

Stellar winds.

The stellar-wind mass-loss rate increases dramatically during the AGB phase, reaching $\sim 10^{-8}$–$\sim 10^{-6}$ $M_\odot$ yr^{-1}; this wind is believed to be powered by radiation pressure on dust forming in the cool outer regions of the photosphere.

6.4.4 Thermal pulsing AGB (TP-AGB); mostly shell H burning [H]

As the helium-burning shell uses up the available helium fuel, its mass decreases; the overlying hydrogen layers contract, heating a shell of hydrogen to the point where it can ignite. Two shell sources now exist, but because their rates of energy generation are so different, the situation is unstable, leading to a phase of *thermal pulsing*. During most of this phase, the energy comes from hydrogen shell burning, with relatively brief helium shell 'flashes'.

Schematically, the cycle proceeds as follows:

1. We start with a degenerate CO core and a helium-burning shell.

2. As the star ascends the AGB, the 'ash' from the helium-burning shell adds mass to the CO core, driving the luminosity up (as described above). The helium-shell mass is reduced (by He burning, and the continuing second dredge-up); when it reaches $\sim 0.02 M_\odot$, the temperature at the H/He interface is sufficient to ignite shell hydrogen burning.

   The structure of the star is now:

   a degenerate CO core;

   a helium-burning shell;

   a hydrogen-burning shell; and

   an outer (hydrogen-rich) envelope.

3. Burning in the helium shell is unstable because of the thinness of the shell, coupled with the strong temperature dependence of the 3α reaction. Essentially, the energy liberated raises the pressure, initiating expansion; once the expansion has progressed far enough, cooling sets in and the helium burning stops (Iben & Renzini 1983).

4. The hydrogen shell continues to burn, increasing the mass of the underlying helium layer between the CO core and the H-burning shell. H-shell burning is providing practically all the luminosity; this phase lasts $\sim 10^3$–10^5 yr, depending on core mass (smaller cores lead to longer cycles).

5. The helium layer between the core and hydrogen-burning shell is mildly degenerate. As mass is added from the hydrogen-burning shell, the pressure and temperature rise in this zone.

6. Eventually, helium is reignited explosively in a 'helium flash' when the temperature reaches $\sim 10^8$ K. The fusion raises the temperature, which increases the reaction rate, which raises the temperature... The helium-burning layer doesn't expand at first because of the degeneracy (pressure support isn't thermal), so there is no regulation of the rate of fusion (hence a 'flash'), but then the temperature reaches the point where the degeneracy is lifted.

7. Considerable energy is generated in the flash ($\sim 10^8 L_\odot$for $\sim$a year), but much of it goes into expanding the helium zone. The helium-burning region stabilises and stable burning ensues for $\sim 10^2$ yr. As a result of expansion and cooling, shell hydrogen burning is extinguished, which restores the conditions at the end of the E-AGB phase (step 1); and the cycle repeats.

The thermal pulse cycle can repeat many times, but is barely noticeable at the surface of the star. The helium shell is dormant for more than 99% of the cycle; hydrogen shell burning is the main luminosity source averaged over the time.

Third dredge-up.

The large energy flux above the helium shell at step 7 above demands convective energy transport. If this merges with the convective outer envelope, the products of helium burning (notably carbon) can be exposed at the surface, in a *third dredge-up*.[3] This can lead to the formation of 'carbon stars' (surface C/O$>$ 1; cp. solar, C/O$\simeq$ 0.4).

However, for sufficiently massive stars ($M \gtrsim$ 4–5$\mathrm{M}_\odot$), the base of the convective envelope can reach $T \gtrsim 5 \times 10^7$ K, hot enough to convert dredged-up carbon to nitrogen through the CN cycle ('hot bottom burning').

Furthermore, ^{14}N is a product of CNO burning in the hydrogen shell. During a thermal pulse, this can be burnt to neon through

$$^{14}\mathrm{N}(\alpha,\gamma)^{18}\mathrm{F}(\beta^+,\nu)^{18}\mathrm{O}(\alpha,\gamma)^{22}\mathrm{Ne}$$

and then, for stars with cores masses $\gtrsim$1$\mathrm{M}_\odot$(i.e., high enough temperatures),

$$^{22}\mathrm{Ne}(\alpha,\mathrm{n})^{25}\mathrm{Mg},$$

generating a flux of neutrons at every thermal pulse. This provides 'fuel' for the *s* process; and the repeated application of neutron fluxes generates heavier and heavier elements. The third dredge-up can (and does) reveal such elements, including, famously, ^{99}Tc, a radioactive isotope[4] with a half-life of only $\sim 2 \times 10^5$ yr.

Mass loss.

TP-AGB stars show dusty, massive, slow stellar winds, with $\dot{M}$ reaching up to $\sim 10^{-4}$ $\mathrm{M}_\odot$ yr^{-1}. Since this can continue for $\sim 10^5$ yr, the star can lose a substantial fraction of its original mass through such winds.

The dust composition depends on the C/O ratio (i.e., on the nature of the third dredge-up). If C/O$<$ 1 (by number), then all the carbon is locked into the stable CO molecule, and the remaining oxygen forms silicate dust; if C/O$>$ 1, then all the oxygen is locked into CO, and the

[3]The dredge-up of helium-burning products is always called a third dredge-up, even if a second dredge-up did not occur; and multiple third dredge-ups can occur in a star.

[4]Technetium was discovered in red-giant spectra by Paul Merrill in 1952.

dust is carbon-rich (containing, e.g., SiC and C_nH_n molecules). The mechanism responsible for the substantial mass loss is not well understood; radiation pressure must play a role, but pulsational instabilities may be involved (many AGB stars are long-period variables).

6.4.5 Post-AGB evolution

The mass of the convective envelope decreases steadily through the AGB phase, through shell burning (removes mass from the bottom of the envelope) and stellar winds (removes mass from the top of the envelope). When the mass of the hydrogen envelope becomes very small ($\sim 10^{-3}$–$10^{-2}\mathrm{M}_\odot$), convection can no longer be sustained and the envelope contracts into radiative equilibrium.

Hydrogen shell burning is still taking place, so the luminosity is unchanged (following the Paczynski relation); the star therefore leaves the AGB, moving leftwards in the HRD. As it gets hotter it ionizes the circumstellar envelope generated principally through AGB mass loss, and develops a fast, radiation-driven wind ($\dot{M} \simeq 10^{-7}$ $\mathrm{M}_\odot$ $\mathrm{yr}^{-1}, v \simeq 2000$ $\mathrm{km\ s}^{-1}$). These two effects produce the diversity of planetary nebulae observed.

When the envelope mass falls to $\sim 10^{-5}\mathrm{M}_\odot$, the H-burning shell dies out, and the remnant core becomes a cooling white dwarf at $\sim$3–10×10^4 K, slowly radiating away its thermal energy over $\sim$a Hubble time.[5]

6.4.6 Summary

To review the key evolutionary stages of an intermediate-mass star:

Main-sequence stars: powered by core hydrogen burning; longest evolutionary phase.

RGB stars: powered by shell hydrogen burning. First dredge-up.

Horizontal-branch stars: powered by shell hydrogen burning and core helium burning. Second-longest evolutionary phase ($\sim$10% of MS lifetime).

Early-AGB stars: powered by shell helium burning. Second dredge-up (of CNO processed material from dormant H-burning shell).

TP-AGB stars: thermally pulsing. Hydrogen shell burning is the main energy source, with repeating brief helium shell flashes (thermal pulses). Third dredge-up (of He-burning products and *s*-process elements).

Planetary nebula; white dwarf.

[5] In some cases a late thermal pulse can bring the star back as a 'born-again' AGB star.