

Q1

a) For distinguishable particles  $Z = z^N$  since we multiply partition functions for different systems - possibilities multiply and extensive quantities must be additive.

In distinguishable particles can be interchanged without changing the state. So then  $Z = z^N$  overcounts the number of distinct states.  $N$  particles can be re-arranged in  $N!$  ways. So to reduce  $\frac{1}{N!}$  is required to remove the overcounting of indistinguishable particles [6]  
 Negligible multiple ~~occupancy~~ occupancy

b)  $F = E - TS$  so  $dF = dE - Tds - SdT$   
 but  $dE = Tds - pdv + \mu dN$   
 so  $dF = -SdT - pdv + \mu dN$

so  $P = - \left. \frac{\partial F}{\partial V} \right|_{T, N}$

Since  $F = -kT \ln Z$

it then follows that  $P = kT \frac{\partial \ln Z}{\partial V}$  [6]

c)  $Z = \frac{1}{N!} z^N$

so  $\ln Z = N \ln z - \ln N!$

$P = kT \frac{\partial \ln Z}{\partial V} = NkT \frac{\partial \ln z}{\partial V}$

$\ln z = \ln V + \ln \left( \frac{m k T}{2 \pi \hbar^2} \right)^{3/2}$

so  $\frac{\partial \ln z}{\partial V} = \frac{\partial \ln V}{\partial V} = \frac{1}{V}$

so  $P = \frac{NkT}{V}$  — which is ideal gas eq<sup>n</sup> at state [6]

- d) Temperature  $T \rightarrow$  thermal energy  $kT$   
 $\rightarrow$  kinetic energy  $\frac{1}{2} m v^2$   
 $\rightarrow$  velocity  $v = \sqrt{\frac{2kT}{m}}$   
 $\rightarrow$  momentum  $mp = \sqrt{2mkT}$   
 $\rightarrow$  wave number  $k = \sqrt{\frac{2m kT}{\hbar^2}}$   
 $\rightarrow$  wave length  $\lambda = \frac{2\pi}{k} = 2\pi \sqrt{\frac{\hbar^2}{2mkT}}$   
 which is almost  $\lambda$ .

[6]

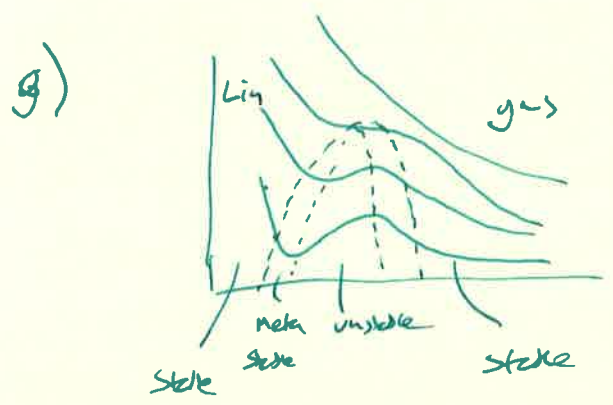
- e) Split interaction into:  
 hard core repulsion  
 weak long-range attraction.

Account for repulsion by simply excluding the volume from the P.F. calculation ( $V_{ex}$ )

Account for attraction by a small mean energy  $\langle E \rangle$

thus 
$$Z = \frac{V}{\Lambda^3} \rightarrow \frac{V - V_{ex}}{\Lambda^3} e^{-\langle E \rangle / kT}$$
 [6]

- f) Many-body problem reduces to a single-particle in a "mean field" [4]



[6]

Q2

a) Oxygen is an 'exchange' interaction  
 consisting of  $\sum \underline{\mu}_1 \cdot \underline{\mu}_2 + \text{the energy}$ .

Since magnetic energy is  $\sum \underline{\mu} \cdot \underline{B}$   
 including exchange, the energy becomes

$$E = \sum \underline{\mu} \cdot \underline{B} + c \sum \underline{\mu}_1 \cdot \underline{\mu}_2$$

$$= \sum \left[ \underline{\mu} \cdot (\underline{B} + c \sum \underline{\mu}) \right] \quad 4$$

And since  $\sum \underline{\mu}$  is the magnetization  $M$   
 we take the average

$$E = \sum \underline{\mu} \cdot (\underline{B} + \underline{b}) \quad 4$$

where

$$\underline{b} = \lambda \underline{M}$$

[8]

b)  $\frac{M}{M_0} = \text{Loh} \frac{\underline{\mu} B}{kT}$   
 $B \rightarrow B_0 + \underline{b}$

$$\frac{M}{M_0} = \text{Loh} \frac{\underline{\mu} (B_0 + \underline{b})}{kT} \quad 3$$

Spontaneous magnetization when  $B_0 = 0$

$$\frac{M}{M_0} = \text{Loh} \frac{\underline{\mu} \underline{b}}{kT} = \text{Loh} \left( \frac{M_0}{N} \frac{\lambda M}{kT} \right)$$

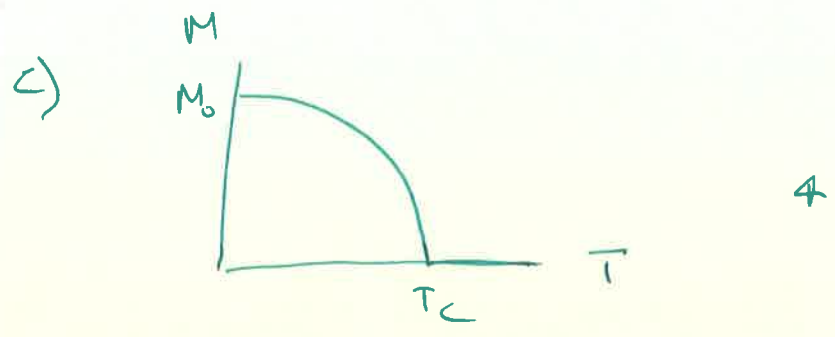
$$= \text{Loh} \left( \frac{M}{M_0} \frac{M_0^2 \lambda}{NkT} \right)$$

$$= \text{Loh} \left( \frac{M}{M_0} \frac{T_c}{T} \right) \quad 3$$

where  $T_c = \lambda M_0^2 / Nk$ .

$T_c$  is temperature at which magnetization vanishes  
 (critical temperature)

[8]



Transition is 2<sup>nd</sup> order  $\rightarrow$  M

gas + zero continuously  $\rightarrow$   $T \rightarrow T_c$   $\neq$  [8]

d)

$$\frac{M}{M_0} = \text{let } \frac{M}{M_0} = \frac{T_c}{T}$$

$$\sim \frac{M}{M_0} \frac{T_c}{T} - \frac{1}{3} \left( \frac{M}{M_0} \frac{T_c}{T} \right)^3$$

$$1 \sim \frac{T_c}{T} - \frac{1}{3} \left( \frac{M}{M_0} \right)^2 \left( \frac{T_c}{T} \right)^3$$

$$\frac{M}{M_0} \sim \sqrt{3} \frac{T}{T_c} \left( 1 - \frac{T}{T_c} \right)^{1/2}$$

$$\sim \sqrt{3} \left( 1 - \frac{T}{T_c} \right)^{1/2} \quad \text{since } T \sim T_c$$

i.e.  $\frac{M}{M_0} \propto \left( 1 - \frac{T}{T_c} \right)^{1/2}$  [8]

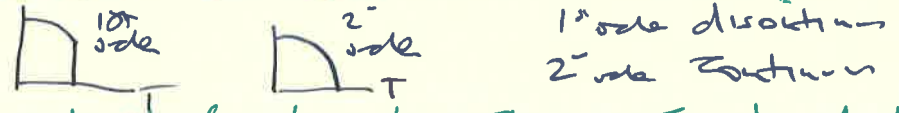
e) Here  $\beta = \frac{1}{2}$ .

Real magnets  $\beta = \frac{1}{3}$   $\neq$

difference is due to neglect of fluctuations at critical point  $\neq$  (mean field model). [8]

Q 3

a) Order Parameter - measure of the order in the order (low-T) phase. Equals zero in the disordered phase & will go to unity for full order (T=0) (usually). [6]

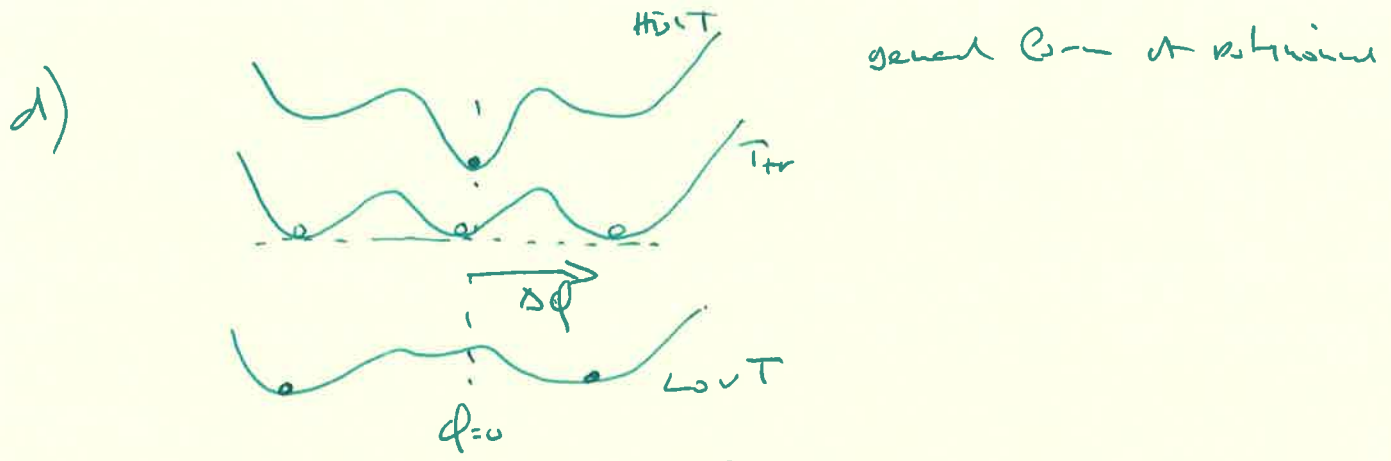


b) i) Structural phase transition - Change of + charge layers displacement from center of - charge [2]

ii) Reflection symmetry. (not really rotational symmetry) [2]

iii) Polarization (not really a vector as the direction is specific). [2]  
i.e. polarization along the displacement direction

c)  $\phi^6$  is just enough terms to give genuine freedom of 1st order transition. 2  
No odd powers because of reflection symmetry 2 [4]



As T drops to middle  $\phi=0$  minimum rises as the side minima drop.  
When  $T=T_{tr}$   $\phi$  can jump between 0 as finite value  $\Rightarrow$  1st order transition [6]

e) Transition begins 2<sup>nd</sup> order  
 when  $F_4$  is positive. In that  
 case  $F_6$  may be ignored and we  
 have to use 2<sup>nd</sup> order case.

[4]

f) Let  $F_0 = 0$

Then  $F = 0$  when  $q = 0$

So require  $F = 0$  & first  $q$  at  
 the transition, so  $F'(0) = 0$ .

~~$$F = F_2 q^2 + F_4 q^4 + F_6 q^6$$~~

$$\text{i.e. } F_2 q^2 + F_4 q^4 + F_6 q^6 = 0$$

$$\text{or } 2F_2 q + 4F_4 q^3 + 6F_6 q^5 = 0$$

$$\text{or } F_2 + F_4 q^2 + F_6 q^4 = 0$$

$$2F_2 + 4F_4 q^2 + 6F_6 q^4 = 0$$

$$\rightarrow 2F_4 q^2 + 4F_6 q^4 = 0$$

$$\text{or } 2F_4 + 4F_6 q^2 = 0 \quad \text{if } q \neq 0$$

$$\text{so } q = \sqrt{\frac{-F_4}{2F_6}}$$

The other value of  $q$  is zero. So the case  
 $q = 0$

$$\Delta q = \sqrt{\frac{-F_4}{2F_6}}$$

[8]

g) Lando says works in the vicinity  
 of a critical point (to the extent  
 that it works at all!!).

i.e.  $q \approx 0$ .

But by definition, in a 1<sup>st</sup> order  
 transition  $q$  is finite. Since  $\tau_{00}$  is  
 a discontinuity in  $q$ .

So it will only be useful if to jump in  $q$

↳ small. i.e. Watch for order transitions. [6]

Q4

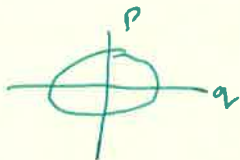
a) Phase space is the space of generalized coordinates & generalized momenta of a system (classical description) 2

In the Boltzmann picture each particle will be represented by a point in 6-dimensional space. So a system of N particles is represented by a 'cloud' of N points in the 6-dim phase space. 2

In the Gibbs picture an N-particle system will have 3N ps & 3N qs, so the phase space will be 6N dimensional - very large. The microstate of the system is represented by a single point in the phase space. 2

[6]

b)



undamped

Simple ellipse

3



damped.

Spirals into center (damped)

3

[6]



c) Coherent expression is

$$S = k \ln P$$

[5]

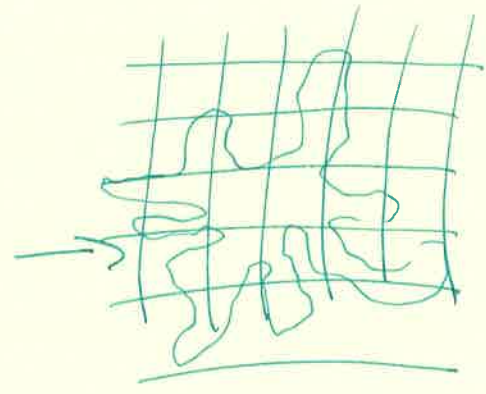
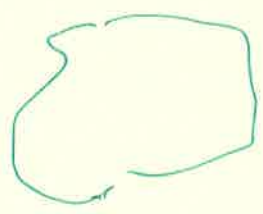
but this hides the question of the size of the cells in phase space.

d) Liouville's theorem:  $P = \text{const}$

2<sup>nd</sup> Law requires  $P$  decreases with time.

Explanation is in the nature of the flow of points in phase space. [7]

e) Resolution

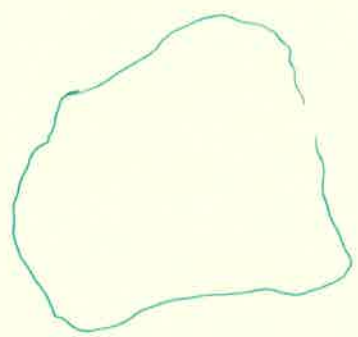


Coarse graining grid

2

which appears as

(Larger volume - smaller density)



2

Coarse graining is based on the idea that there is a volume in phase space finer than which you cannot resolve.

- This can be regarded as a statement of the uncertainty principle.

[8]

f) This Law refers to the uniqueness of the quantum ground state. (Microstates at  $T=0$ ). So  $S = k_B \ln \Omega = 0$  ( $\Omega=1$ ). (Simple argument).

If classical mechanics were to apply then at  $T=0$  there would be a single point in phase space —  $p$  would be a  $\delta$  function and then  $S$  would be  $-\infty$ . [2]

# Q5

## Landau - Einstein Views :

- \* The L+L book on Statistical Physics derives all A macroscopic thermodynamics from microscopic principles
- \* Einstein, in his debates with others on the foundations of Quantum Mechanics, often used purely thermodynamic arguments.
- \* The Landau view is pure reductionism. This is exemplified by much of Boltzmann's work on thermodynamics - and, of course, strongly criticised by Ernst Mach.
- \* Arguments from pure classical thermodynamics would physically plausible that are independent of microscopic structure. In this respect one may say that the properties that can be related using classical thermodynamics are universal, independent of the microscopic, whereas those which cannot are system-specific. An elegant example is Onsager's rule for the viscosity of a transport coefficient - such as for thermoelectricity.

## 'Perverse' behaviour

- \* Einstein was much inspired by Quantum Mechanics - an ultimate in microscopic
- \* The Einstein model of heat capacity is a good example of this.
- \* BEC is also a possible example
- \* Landau was the great 'Model builder' - nothing so far from the reductionist approach.
- \* The Landau theory of Phase Transitions is a great example where one specifically eschews the microscopic.
- \* He used the concept of the "effective Hamiltonian" : an up-front statement that he was not interested in the ultimate microscopic description.

# Emergence

- \* The basic idea of Emergence is that different descriptions are appropriate for different scales.
- \* Thus one talks of the temperature of a collection of electrons, but the temperature of a single electron is meaningless.
- \* The behavior at the macroscopic level must be consistent with the microscopic - but not necessarily derivable from it.
- \* Indeed one expects the macroscopic description to be robust against microscopic changes.
- \* Example: thermal equilibrium requires interactions but the nature of the thermal equilibrium does not depend on the precise nature of the (small) interactions.