PH4211 Statistical Mechanics

Problem Sheet 5

5.1 A random quantity has an exponential autocorrelation function $G(t) = G(0)e^{-\gamma t}$. Calculate its correlation time using the usual definition.

5.2 Show that the autocorrelation function of a periodically varying quantity $m(t) = m\cos\omega t$ is given by

$$G(t) = \frac{m^2}{2}\cos\omega t .$$

Show that the autocorrelation function is independent of the *phase* of $m(t)$. In other words, show that if $m(t) = m\cos(\omega t + \varphi)$, then $G(t)$ is independent of φ.

5.3 The dynamical response function $X(t)$ must vanish at zero times, as shown in Fig. 5.13. What is the physical explanation of this? What is the consequence for the step response function $\Phi(t)$? Is this compatible with an exponentially decaying $\Phi(t)$?

5.4 In Section 5.3 we examined the form of the dynamical susceptibility $\chi(\omega)$ that followed from the assumption that the step response function $\Phi(t)$ decayed exponentially. In this question consider a step response function that decays with a gaussian profile, $\Phi(t) = \chi_0 e^{-t^2/2\tau^2}$. Evaluate the real and imaginary parts of the dynamical susceptibility and plot them as a function of frequency. The real part of the susceptibility is difficult to evaluate without a symbolic mathematics system such as *Mathematica*. Compare and discuss the differences and similarities between this susceptibility and that deduced from the exponential step response function (Debye susceptibility).

5.5 The Debye form for the dynamical susceptibility is

$$\chi'(\omega) = \chi_0 \frac{1}{1+\omega^2\tau^2}$$

$$\chi''(\omega) = \chi_0 \frac{\omega\tau}{1+\omega^2\tau^2}.$$

Plot the real part against the imaginary part and show that the figure corresponds to a semicircle. This is known as a Cole-Cole plot.

5.6 Plot the Cole-Cole plot (Problem 5.5) for the dynamical susceptibility considered in Problem 5.4. How does it differ from that of the Debye susceptibility.

5.7 The full quantum-mechanical calculation of the Johnson noise of a resistor gives

$$\left\langle v^2 \right\rangle_{\Delta f} = 4R\frac{hf}{e^{hf/kT}-1}\Delta f .$$

Show that this reduces to the classical Nyquist expression at low frequencies. At what frequency will there start to be serious deviations from the Nyquist value? Estimate the value of this frequency.

5.8 Show that for the Debye susceptibility, the relation

$$\chi_0 = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\chi''(\omega)}{\omega}\,\mathrm{d}\omega$$

holds. Demonstrate that χ'' vanishes sufficiently fast at $\omega = 0$ so there is no pole in the integral and there is thus no need to take the principal part of the integral in the Kramers-Kronig relations.

5.9 In Section 5.3.6 we considered an electrical analogue of the Langevin Equation based on a circuit comprising an inductor and a resistor. In this problem we shall examine a different analogue: a circuit of a capacitor and a resistor. Show that the equation analogous to the Langevin equation, in this case, is

$$C\frac{\mathrm{d}V(t)}{\mathrm{d}t} + \frac{1}{R}V(t) = I(t).$$

Hence show that the fluctuation-dissipation result relates the resistance to the current fluctuations through

$$\frac{1}{R} = \frac{1}{2kT}\int_{-\infty}^{\infty}\langle I(0)I(t)\rangle\,\mathrm{d}t.$$