PH4211 Statistical Mechanics

Problem Sheet 2

2.1 In Section 2.1 we saw that the density of free-particle states for a three dimensional volume V was shown to be

$$g(\varepsilon) = \frac{1}{4}\frac{V}{\pi^2\hbar^3}(2m)^{3/2}\,\varepsilon^{1/2}\,.$$

This followed from counting the number of states in the octant of radius

$$R = \sqrt{n_x^2 + n_y^2 + n_z^2}\,.$$

By similar arguments show that in two dimensions, by counting the number of states in the quadrant of radius

$$R = \sqrt{n_x^2 + n_y^2}\,,$$

the density of states is given by

$$g(\varepsilon) = \frac{mA}{2\pi\hbar^2}$$

where A is the area. Note in two dimensions the density of states is independent of energy.

And similarly, show that in one dimension the density of states is

$$g(\varepsilon) = \frac{L}{\pi\hbar}\left(\frac{m}{2}\right)^{1/2}\varepsilon^{-1/2}\,.$$

2.2 In Sections 2.3.1 and 2.3.2 the ideal gas partition function was calculated quantum-mechanically and classically. Although the calculations were quite different, they both resulted in (different) Gaussian integrals. By writing the Gaussian integral of the classical case as

$$\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}z\,\,e^{-\left(x^2+y^2+z^2\right)}$$

and transforming to spherical polar coordinates, you can perform the integration over θ and φ trivially. Show that the remaining integral can be reduced to that of the quantum case.

2.3 The Sakur-Tetrode equation, discussed in Section 2.3.3,

$$S = Nk\ln V - Nk\ln N + \frac{3}{2}Nk\ln T + Nks_0$$

is often interpreted as indicating different contributions to the entropy: the volume contribution is in the first term, the number contribution in the second term and the temperature contribution in the third term. Show that such an identification is fallacious, by demonstrating that the various contributions depend on the choice of units adopted – even though the total sum is independent. Discuss the origin of the fallacy.

2.4 Show that the Fermi energy for a two-dimensional gas of Fermions is

$$\varepsilon_F = \frac{2\pi\hbar^2}{\alpha m}\frac{N}{A}$$

where A is the area of the system.

2.5 Show that the chemical potential of a two-dimensional gas of fermions may be expressed analytically as

$$\mu = kT \ln\left\{e^{\varepsilon_F/kT} - 1\right\}$$

2.6 Calculate the low temperature chemical potential of a two-dimensional gas of fermions by the Sommerfeld expansion method of Section 2.4.3. Observe that the temperature series expansion terminates. Compare this result with the exact result of the previous question. Discuss the difference between the two results.

2.7 The general formula for the Fermi integrals I_n of Section 2.4.3 was quoted as

$$I_n = \int_{-\infty}^{\infty} \frac{e^x}{\left(e^x + 1\right)^2} x^n \, dx$$

$$= \left(2 - 2^{2-n}\right)\zeta(n)n!$$

Derive this result. (You might find the discussion in Landau and Lifshitz, Statistical Physics, helpful.)

2.8 Obtain the chemical potential μ, the internal energy E and the heat capacity C_v for system with general density of states $g(\varepsilon)$ as in Section 2.4.4. I.e. show that these are given in terms of the behaviour of the density of states at the Fermi surface.

2.9 Consider the Bose gas at low temperatures. You saw in Section 2.5.2 and 2.5.3 that when the occupation of the ground state is appreciable then the chemical potential μ is very small and it may be ignored, compared with ε in the integral for the number of excited states.

Show that when the ground state occupation N_0 is appreciable then μ may be approximated by

$$\mu \sim -kT/N_0 \, .$$

Now consider the more stringent requirement that μ may be neglected in comparison with ε in the integral for the number of excited states. This will be satisfied if μ is much less than the energy ε_1 of the first excited state. The expression for ε_1 is

$$\varepsilon_1 \sim \frac{\pi^2\hbar^2}{2mV^{2/3}} \, .$$

Where does this expression come from?

Show that the condition $\mu \ll \varepsilon_1$ is satisfied when $N \gg 10^{15}$ (approximately) when considering 1cm^3 of ^{4}He (molar volume 27cm^3) at a temperature of about 1K.

Thus show that the expression

$$N_0 = N \left\{ 1 - \left(\frac{T}{T_c} \right)^{3/2} \right\}$$

is then valid to temperatures below T_c right up to within $\sim 10^{-8} T_c$ of the critical temperature.

2.10 Liquid ^{4}He has a molar volume at saturated vapour pressure of 27cm^3. Treating the liquid as an ideal gas of bosons, find the temperature at which Bose-Einstein condensation will occur. How will this temperature change as the pressure on the fluid is increased?

2.11 The superfluid transition temperature of liquid helium decreases with increasing pressure. Very approximately $\partial T_c / \partial p \sim -0.015 \text{ K bar}^{-1}$. How does this compare with the behaviour predicted from the Bose-Einstein condensation?

2.12 Show that below the transition temperature the entropy of a Bose gas is given by
$$S = \frac{5}{3} Nk \frac{I_{3/2}}{I_{1/2}} \left(\frac{T}{T_c} \right)^{3/2}.$$
Since the number of excited particles is given by
$$N_{ex} = N \left(\frac{T}{T_c} \right)^{3/2},$$
show that the entropy per excited particle is given by
$$\frac{S}{N_{ex}} = \frac{5}{3} \frac{I_{3/2}}{I_{1/2}} k \approx 1.28k.$$
Discuss the connection between this result and the two fluid model of superfluid ^{4}He.

2.13 Show that the Bose-Einstein transition temperature of a gas of bosons and the Fermi temperature for a gas of 'similar' fermions are of comparable magnitude. Discuss why this should be.

2.14 In Section 2.6 we studied a paramagnetic *solid*: a collection of essentially *distinguishable* magnetic moments. If we were to consider a (classical) gas of indistinguishable magnetic moments, how would the partition function be modified? What would the observable consequences of this modification?

2.15 Show, using arguments similar to those in Section 2.1.3, that the energy levels of an ultra-relativistic or a massless particle with energy-momentum relation $E = cp$ are given by
$$\varepsilon = \frac{c\pi\hbar}{V^{1/3}} \left(n_x^2 + n_y^2 + n_z^2 \right)^{1/2}.$$
Hence show that the pressure of a gas of such particles is one third of the (internal) energy density.

2.16 Evaluate the Fermi temperature for liquid ^{3}He, assuming it to be a Fermi "gas". Its molar volume is 36 cm^3. Calculate the de Broglie wavelength at $T = T_F$ and show that it is comparable to the interparticle spacing as expected.

2.17 In Problem 2.1 we found the expression for the energy density of states $g(\varepsilon)$ for a gas of fermions confined to two dimensions and we saw that it was independent of energy. What surface density of electrons is necessary in order that $T_F = 100$ mK? Show that, for a given area, the heat capacity is independent of the number of electrons.

2.18 Use the Sommerfeld expansion method of Section 2.4.3 to show that the Fermi-Dirac distribution function may be approximated, at low temperatures, by

$$\frac{1}{e^{(\varepsilon-\mu)/kT}+1} \sim \Theta\left(\mu-\varepsilon\right)-\frac{\pi^2}{6}\left(kT\right)^2 \delta'\left(\varepsilon-\mu\right)+\ldots$$

where Θ is the unit step function and δ' is the first derivative of the Dirac delta function.

Can you write down the general term of the series?