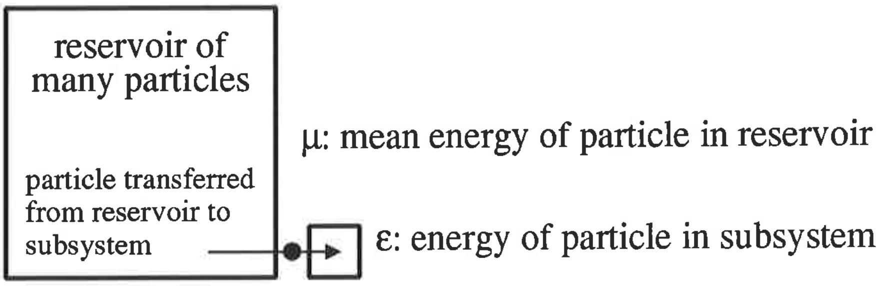
1.5.2 The quantum distribution functions

We shall obtain the distribution functions for particles obeying Fermi-Dirac statistics and those obeying Bose-Einstein statistics. Thus we want to know the mean number of particles which may be found in a given quantum state. The method is based on an idea in Feynman's book *Statistical Mechanics*, Benjamin (1972). We start by considering an idealised model, of a subsystem comprising a single quantum state of energy ε, in thermal equilibrium with a reservoir of many particles. The mean energy of a particle in the reservoir is denoted by μ (we will tighten up on the precise definition of *mean energy* later).



Transfer of a particle from reservoir to the subsystem

A particle may be in the reservoir or may be in the subsystem. The probability that it is in the subsystem is proportional to the Boltzmann factor $\exp(-\varepsilon / kT)$, while the probability that it is in the reservoir is proportional to $\exp(-\mu / kT)$. If $P(1)$ is the probability that there is one particle in the subsystem and $P(0)$ is the probability of no particles in the subsystem, then we may write

$$\frac{P(1)}{P(0)} = \exp-\frac{(\varepsilon - \mu)}{kT} \quad \text{or} \quad P(1) = P(0)\exp-\frac{(\varepsilon - \mu)}{kT}.$$

If the statistics allow (for Bosons, but not for Fermions) then we may transfer more particles from the reservoir to the subsystem. Each particle transferred will lose an energy μ and gain an energy ε. Associated with the transfer of n particles there will therefore be a Boltzmann factor of $\exp -n(\varepsilon - \mu)/kT$. And so the probability of having n particles in the subsystem is

$$P(n) = P(0)\exp-\frac{n(\varepsilon - \mu)}{kT}. \tag{1}$$

Let us put

$$x = \exp-\frac{(\varepsilon - \mu)}{kT}. \tag{2}$$

Then

$$P(n) = P(0)x^n. \tag{3}$$

Normalisation requires that all possible probabilities sum to unity. For Fermions we know that n can take on only the values 0 and 1, while for Bosons n can be any integer. Thus we have

$$P(0) + P(1) = 1 \qquad \text{for Fermions}$$

$$\sum_{n=0}^{\infty} P(n) = 1 \qquad \text{for Bosons}$$

which can be written, quite generally as

$$\sum_{n=0}^{a} P(n) = 1 \tag{4}$$

where $a = 1$ for Fermions and $a = \infty$ for Bosons.

Since $P(n)$ is given by Equation (3), the normalisation requirement may be expressed as

$$P(0) \sum_{n=0}^{a} x^n = 1$$

which gives us $P(0)$:

$$P(0) = \left\{\sum_{n=0}^{a} x^n\right\}^{-1}.$$

We will be encountering the above sum of powers of x quite frequently, so let's denote it by the symbol Σ:

$$\Sigma = \sum_{n=0}^{a} x^n. \tag{5}$$

In terms of this

$$P(0) = \Sigma^{-1}, \tag{6}$$

and then from Equation (3)

$$P(n) = x^n / \Sigma. \tag{7}$$

What we want to know is the *mean* number of particles in the subsystem. That is, we want to calculate

$$\bar{n} = \sum_{n=0}^{a} nP(n), \tag{8}$$

which is given by

$$\bar{n} = \frac{1}{\Sigma} \sum_{n=0}^{a} nx^n. \tag{9}$$

The sum of nx^n may be found by using a trick (which is really at the heart of many Statistical Mechanics calculations). The sum differs from the previous sum Σ which we used, because of the extra factor of n. Now we can bring down an n from x^n by differentiation. Thus we write

$$nx^n = x\frac{\mathrm{d}}{\mathrm{d}x}x^n,$$

so that

$$\sum_{n=0}^{a} nx^n = x\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{a} x^n.$$

Observe that the sum on the right hand side here is our original sum Σ. This means that $\bar{n}$ can be expressed as

$$\bar{n} = x\frac{1}{\Sigma}\frac{\mathrm{d}\Sigma}{\mathrm{d}x}$$

or

$$\bar{n} = x\frac{\mathrm{d}\ln\Sigma}{\mathrm{d}x}.$$

It remains, then, to evaluate Σ for the two cases. For Fermions we know that $a = 1$, so that the sum in Equation (5) is $1 + x$. For Bosons a is infinity; the sum is an infinite (convergent) geometric series. The sum of such a geometric progression is $1/(1 - x)$. Thus we have

Fermions		**Bosons**
$\Sigma = 1 + x$		$\Sigma = (1 - x)^{-1}$
$\ln\Sigma = \ln(1 + x)$		$\ln\Sigma = -\ln(1 - x)$
	upon differentiating	
$\frac{\mathrm{d}\ln\Sigma}{\mathrm{d}x} = \frac{1}{1 + x}$		$\frac{\mathrm{d}\ln\Sigma}{\mathrm{d}x} = \frac{1}{1 - x}$
	so that	
$x\frac{\mathrm{d}\ln\Sigma}{\mathrm{d}x} = \frac{x}{1 + x}$		$x\frac{\mathrm{d}\ln\Sigma}{\mathrm{d}x} = \frac{x}{1 - x}$
	and $\bar{n}$ is then given by	
$\bar{n} = \frac{1}{x^{-1} + 1}$		$\bar{n} = \frac{1}{x^{-1} - 1}$
	Finally, substituting for x from Eq (2):	
$\bar{n} = \frac{1}{\exp(\varepsilon - \mu)/kT + 1}$		$\bar{n} = \frac{1}{\exp(\varepsilon - \mu)/kT - 1}$

These expressions will be recognised as the Fermi-Dirac and the Bose-Einstein distribution functions. However, it is necessary to understand the way in which this idealised model relates to realistic assemblies of Bosons or Fermions. We have focussed attention on a given quantum state, and treated it as if it were apart from the reservoir. In reality the reservoir is the entire system and the quantum state of interest is *in* that system. The entire analysis then follows through so long as the mean energy of a particle, μ, in the system is changed by a negligible amount if a single quantum state is excluded. And this must be so for any macroscopic system.

We now turn to an examination of the meaning of μ within the spirit of this picture. We said that it was the mean energy lost when a particle is removed from the reservoir, which we now understand to mean the entire system. When a particle is removed the system remains otherwise unchanged. In particular the distribution of particles in the other energy states is unchanged - the entropy remains constant. Also the energy of the various states is unchanged as the volume remains constant. Thus our μ is equal to $\partial E/\partial N$ at constant S and V, which when compared with the extended statement of the First Law of Thermodynamics:

$$\mathrm{d}E = T\mathrm{d}S - p\mathrm{d}V + \mu\mathrm{d}N,$$

indicates that our μ corresponds to the conventional definition of the chemical potential.

__________ End of lecture 6