1 Relative maxima, relative minima and saddle points

The developments of the previous section (Multivariate Calculus (part 1)) are helpful in studying maxima and minima of functions of several variables. We restrict our attention here to functions $f(x, y)$ of two variables.

If we define

$$\Delta f(x, y) = f(x + h, y + k) - f(x, y) \tag{1}$$

we say that f has a relative minimum at $P(x_0, y_0)$ if $\Delta f(x_0, y_0) \geq 0$ for all sufficiently small permissible h and k and that f has a relative maximum at P if instead $\Delta f(x_0, y_0) \leq 0$ for all such h and k.

If the point P is an interior point of a region in which f, f_x and f_y exist, Equation (35) of the Section (Multivariate Calculus (part 1)) shows that a necessary condition that f assume a relative maximum or minimum at x_0, y_0 is that

$$f_x = f_y = 0 \quad \text{at} \quad (x_0, y_0). \tag{2}$$

For, when h and k are sufficiently small, the sign of $\Delta f(x_0, y_0)$ will be the same as the sign of $hf_x(x_0, y_0) + kf_y(x_0, y_0)$ when this quantity is not zero, and clearly the sign of this quantity will change as the signs of h and k change unless (2) holds.

Suppose now that the condition (2) is satisfied at a certain point P. Then from Equation. (35) of the Section (Multivariate Calculus (part 1)) we have

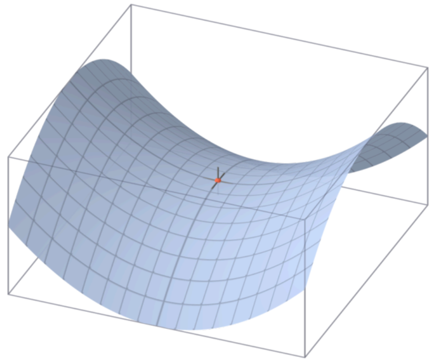
$$\text{sign}[\Delta f(x_0, y_0)] = \text{sign}[h^2 f_{xx}(x_0, y_0) + 2hkf_{xy}(x_0, y_0) + k^2 f_{yy}(x_0, y_0)] \tag{3}$$

when h and k are sufficiently small, unless the bracketed quantity is zero. That quantity is a quadratic expression in h and k of the form $Ah^2 + 2Bhk + Ck^2$. When the discriminant $B^2 - AC$ is positive, and only in that case, there will be two distinct values of the ration h/k for which the expression is zero, the expression having one sign for intermediate values of h/k and the oppsite sign for all other values. Hence a necessary condition that f have either a relative maximum or a relative minimum at $P(x_0, y_0)$ is that

$$\delta = f_{xx} f_{yy} - f_{xy}^2 \geq 0 \quad \text{at} \quad (x_0, y_0). \tag{4}$$

If $\delta < 0$ at a point $P(x_0, y_0)$ where (2) is satisfied, then Δf is positive for some h and k and negative for others. Such a point is called a saddle point. A typical sketch of a saddle is shown in the figure below.

If $\delta > 0$ at (x_0, y_0), then clearly f_{xx} and f_{yy} must be either both positive or both negative at that point. Since $\Delta f(x_0, y_0)$ is of constant sign in either case, when h and k are sufficiently small, it follows from (3)) that the former case corresponds to a realative minimum $\Delta > 0$ and the latter to a relative maximum $\Delta < 0$.



We can summarise the results as follows.
If $f_x = 0$ and $f_y = 0$ at a point P, then at that point f has
(a) a relative maximum if $f_{xx} < 0$ and $f_{xx}f_{yy} > f_{xy}^2$ at P.
(b) a relative minimum if $f_{xx} > 0$ and $f_{xx}f_{yy} > f_{xy}^2$ at P.
(c) a saddle point if $f_{xx}f_{yy} < f_{xy}^2$
When $f_{xx}f_{yy} = f_{xy}^2$ at P, further investigation is necessary.
Points at which $f_x = f_y = 0$ are called critical points.

Example

Locate the critical points of the function $f(x, y) = y^2 - xy + x^2 - 2y + x$ and classify them as relative minimum, relative maximum and saddle points.

$$f_x = -y + 2x + 1 \tag{5}$$

$$`f_y = 2y - x - 2 \tag{6}$$

Equating (5) and (6) to zero, gives the critical point $(0, 1)$.

$$f_{xx} = 2 \tag{7}$$

$$f_{yy} = 2 \tag{8}$$

$$f_{xy} = -1 \tag{9}$$

Therefore

$$\delta = 3 > 0 \tag{10}$$

and

$$f_{xx} > 0 \tag{11}$$

We are in case (b) and $(0, 1)$ is a relative minimum.

Example

Locate the critical points of the function $f(x, y) = x^2y^2 - x^2 - y^2$ and classify them as relative minimum, relative maximum and saddle points.

$$f_x = 2xy^2 - 2x \tag{12}$$

$$`f_y = 2x^2y - 2y \tag{13}$$

Equating (12) and (13) to zero, gives the critical points $(0, 0)$, $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$.

$$f_{xx} = 2y^2 - 2 \tag{14}$$

$$f_{yy} = 2x^2 - 2 \tag{15}$$

$$f_{xy} = 4xy \tag{16}$$

Consider first $(0,0)$. Then $\delta = 4 > 0$ and $f_{xx} = -2$. Therefore we are in case (a) and $(0,0)$ is a relative maximum.

For the critical points $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$, $\delta = -16 < 0$. Therefore we are in case (c) and there are all saddle points.

Example

Locate the critical points of the function $f(x,y) = x^2 + 2bxy + y^2$ and classify them as relative minimum, relative maximum and saddle points.

Answer: Minimum at $(0,0)$ if $b^2 < 1$, saddle point at $(0,0)$ if $b^2 > 1$, minimum along line $y = -x$ if $b = 1$, minimum along line $y = x$ if $b = -1$.