1 Iterated Integrals and Area

Definition of an Iterated Integral

Just as we can take a partial derivative by considering only one of the variables a true variable and holding the rest of the variables constant, we can take a "partial integral". We indicate which is the true variable by writing dx, dy, etc. Also as with partial derivatives, we can take two "partial integrals" taking one variable at a time. In practice, we will either take x first then y or y first then x. We call this an iterated integral or a double integral.

Definition of a Double Integral

Let $f(x, y)$ be a function of two variables defined on a region R bounded below and above by

$$y = g_1(x) \qquad \text{and} \qquad y = g_2(x) \tag{1}$$

and to the left and right by

$$x = a \qquad \text{and} \qquad x = b \tag{2}$$

then the double integral (or iterated integral) of $f(x, y)$ over R is defined by

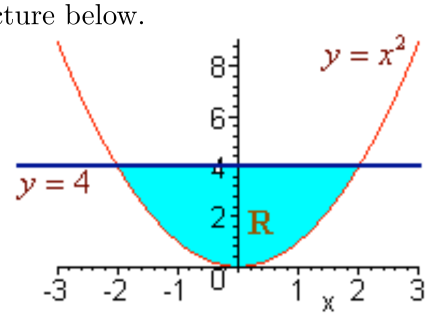
$$\int\int_R f(x, y)dydx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)dydx = \int_a^b [\int_{g_1(x)}^{g_2(x)} f(x, y)dy]dx \tag{3}$$

Example

Find the double integral of $f(x, y) = 6x^2 + 2y$ over R where R is the region between $y = x^2$ and $y = 4$.

Solution

First we have that the inside limits of integration are x^2 and 4. The region is bounded from the left by $x = -2$ and from the right by $x = 2$ as indicated by the picture below.



We now integrate

$$\int_{-2}^{2}\int_{x^2}^{4}(6x^2+2y)dydx = \int_{-2}^{2}[6x^2y+y^2]_{x^2}^{4}dx =$$

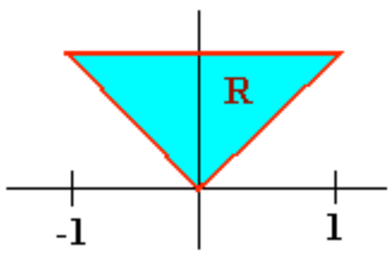
$$= \int_{-2}^{2}(24x^2+16)-(6x^4+x^4)dx = [8x^3+16x-\frac{7}{5}x^5]_{-2}^{2} = 102.4 \qquad (4)$$

2 Changing the Order of Integration

If a region is bounded from the left by $x = h_1(y)$ and the right by $x = h_2(y)$ and below and above by $y = c$ and $y = d$, then we can find the double integral of $dxdy$ by first integrating with respect to x then with respect to y. Sometimes there is a choice to make as to whether to integrate first with respect to x and then with respect to y. We do whatever is easier.

Example

Find the double integral of $f(x, y) = 3y$ over the triangle with vertices (-1,1), (0,0), and (1,1).



Solution

We integrate with respect to x first. The region is bounded on the left and the right by $x = -y$ and $x = y$. The lowest the region gets is $y = 0$ and the highest is $y = 1$. The integral is

$$\int_0^1\int_{-y}^{y}3ydxdy = \int_0^1[3xy]_{-y}^{y}dy = \int_0^1 6y^2dy = [2y^3]_0^1 = 2 \qquad (5)$$

If we try instead to integrate with respect to y first, we have to cut the region into two pieces and perform two iterated integrals. This gives

$$\int_{-1}^{0}\int_{-x}^{1}3ydydx + \int_0^1\int_x^1 3ydydx =$$

$$\int_{-1}^{0}[\frac{3y^2}{2}]_{-x}^{1}dx + \int_0^1[\frac{3y^2}{2}]_x^1dx =$$

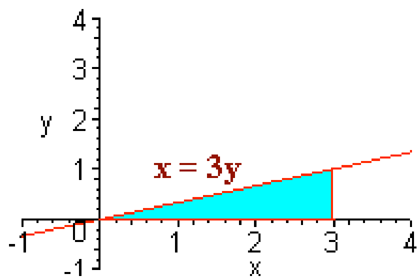
$$\int_{-1}^{0}(\frac{3}{2}-\frac{3}{2}x^2)dx+\int_{0}^{1}(\frac{3}{2}-\frac{3}{2}x^2)dx=$$

$$[\frac{3}{2}x-\frac{x^3}{2}]_{-1}^{0}+[\frac{3}{2}x-\frac{x^3}{2}]_{0}^{1}=2 \qquad (6)$$

Clearly in this example it is easier to integrate with respect to x first!
Example
Evaluate the integral

$$\int_0^1\int_{3y}^3 e^{x^2}dxdy \qquad (7)$$

Solution
Try as you may, you will not find an antiderivative of e^{x^2}. We have another choice. The picture below shows the region.



We can switch the order of integration. The region is bounded above and below by $y = 1/3x$ and $y = 0$. The double integral with respect to y first and then with respect to x is

$$\int_0^3\int_0^{x/3} e^{x^2}dydx \qquad (8)$$

The integrand is just a constant with respect to y so we get

$$\int_0^3 [e^{x^2}y]_0^{x/3}dx=\int_0^3\frac{x}{3}e^{x^2}dx \qquad (9)$$

This integral can be performed with simple u-substitution.

$$u = x^2 \quad du = 2xdx \qquad (10)$$

and the integral becomes

$$\frac{1}{6}\int_0^9 e^u du=[\frac{1}{6}e^u]_0^9=\frac{1}{6}e^9-\frac{1}{6} \qquad (11)$$