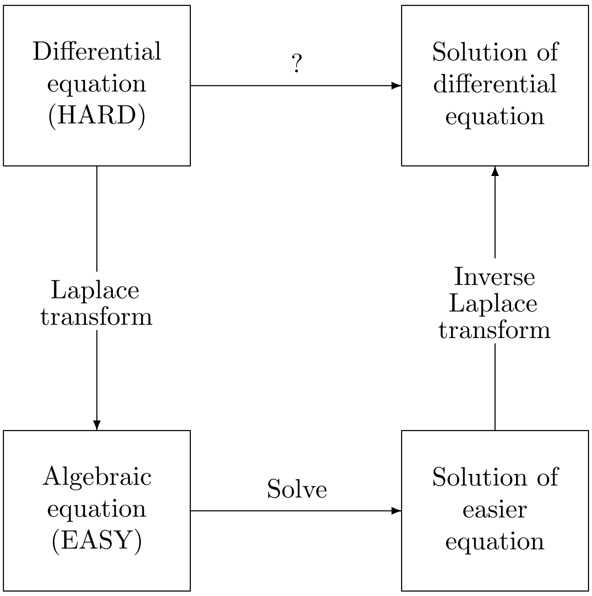
**Laplace transforms**

*This chapter is based on notes written by Dr Helen Wilson*

### 0.0.1 Introduction

The Laplace transform, like the Fourier series, gives us a method of tackling differential equation problems we couldn't otherwise solve.




### 0.0.2 Preparation for Laplace transforms: Improper integrals

Let us look at this integral:

$$I = \int_0^\infty e^{-st}\,\mathrm{d}t$$

It has infinity as its top limit, which means it is called an **improper integral**. Strictly what we mean by this is

$$\begin{aligned} I &= \lim_{M\to\infty}\int_0^M e^{-st}\,\mathrm{d}t = \lim_{M\to\infty}\left[-\frac{e^{-st}}{s}\right]_{t=0}^{M} \\ &= \lim_{M\to\infty}\left[-\frac{e^{-sM}}{s}+\frac{1}{s}\right] = \frac{1}{s} - \lim_{M\to\infty}\left[\frac{e^{-sM}}{s}\right] \end{aligned}$$

Now as $M \to \infty$, $e^{-sM} \to 0$ as long as $s > 0$. If $s = 0$ the integral is not defined because of the $1/s$ (and in fact we're integrating 1 from 0 to $\infty$ which is infinite); if $s < 0$ then $e^{-sM}$ grows as $M \to \infty$ so $I$ does not converge.

So, provided $s > 0$,

$$I = \frac{1}{s}.$$

Now if we let $s$ vary then $I$ is a function of $s$.

### 0.0.3 Definitions: Laplace transform

The Laplace transform of a function $f(t)$ defined on $0 < t < \infty$ is given by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t = F(s)$$

We take a **known function** $f$ of the **original variable** $t$ and create the **Laplace tranform** $\mathcal{L}\{f(t)\}$ which is a new function $F(s)$ of the **transform variable** $s$.

#### Notation

$$\mathcal{L}\{w(t)\} = W(s) \qquad \mathcal{L}\{x(t)\} = X(s)$$

As we saw in the very simple example above, there is often a restriction on $s$ (usually $s > 0$ but sometimes even larger) so that the integral exists.

Before we start finding our first few Laplace transforms, we will devise a set of rules, based on the definition, which will help us to find the Laplace transform of complicated functions.

### 0.0.4 Rules

Throughout these rules, we will assume that we know one Laplace transform already:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,\mathrm{d}t.$$

#### Rule 1: Multiplying by a constant

If we multiply our function by a constant, $a$:

$$\mathcal{L}\{af(t)\} = \int_0^\infty e^{-st} af(t)\,\mathrm{d}t = a\int_0^\infty e^{-st} f(t)\,\mathrm{d}t = aF(s)$$

#### Rule 2: Adding functions together

If we have two answers already:

$$\mathcal{L}\{f_1(t)\} = F_1(s) = \int_0^\infty e^{-st} f_1(t)\,\mathrm{d}t \qquad \text{and} \qquad \mathcal{L}\{f_2(t)\} = F_2(s) = \int_0^\infty e^{-st} f_2(t)\,\mathrm{d}t$$

then:

$$\begin{aligned}
\mathcal{L}\{af_1(t) + bf_2(t)\} &= \int_0^\infty e^{-st}\{af_1(t) + bf_2(t)\}\,\mathrm{d}t = a\int_0^\infty e^{-st} f_1(t)\,\mathrm{d}t + b\int_0^\infty e^{-st} f_2(t)\,\mathrm{d}t \\
\mathcal{L}\{af_1(t) + bf_2(t)\} &= aF_1(s) + bF_2(s)
\end{aligned}$$

#### Rule 3: Multiplying by $e^{at}$

We will leave the proof of this one for homework:

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a).$$

**Rule 4: Differentiation**

We'll use the notation

$$f'(t) = \frac{\mathrm{d}f}{\mathrm{d}t}$$

What is the Laplace transform of the derivative?

$$\mathcal{L}\{f'(t)\} \quad = \quad \int_0^\infty e^{-st}\frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t$$

We can integrate by parts

$$u = e^{-st} \qquad \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}t}$$
$$\frac{\mathrm{d}u}{\mathrm{d}t} = -se^{-st} \qquad v = f(t)$$

$$\begin{aligned}\mathcal{L}\{f'(t)\} &= [e^{-st}f(t)]_{t=0}^{\infty} - \int_0^\infty -se^{-st}f(t)\,\mathrm{d}t = \{0 - f(0)\} + s\int_0^\infty e^{-st}f(t)\,\mathrm{d}t \\ \mathcal{L}\{f'(t)\} &= sF(s) - f(0)\end{aligned}$$

**Rule 5: second derivative**

Again, the proof is left for homework:

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

**Rule 6: Multiply by $-t$**

$$\begin{aligned}\mathcal{L}\{-tf(t)\} &= \int_0^\infty e^{-st}(-tf(t))\,\mathrm{d}t = \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s}\left\{e^{-st}f(t)\right\}\,\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}s}\left\{\int_0^\infty e^{-st}f(t)\,\mathrm{d}t\right\} \\ \mathcal{L}\{-tf(t)\} &= \frac{\mathrm{d}}{\mathrm{d}s}F(s)\end{aligned}$$

**Rule 7: Multiply by $t^2$**

Again, the proof is left for homework:

$$\mathcal{L}\{t^2f(t)\} = \frac{\mathrm{d}^2F(s)}{\mathrm{d}s^2}.$$

### 0.0.5 Common functions

**Result 1** $f(t) = 1$. We've seen this one, when we looked at improper integrals:

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}1\,\mathrm{d}t = \frac{1}{s}$$

**Result 2** $f(t) = t$. We will carry out the improper integral carefully:

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}t\,\mathrm{d}t = \lim_{M\to\infty}\int_0^M te^{-st}\,\mathrm{d}t$$

We integrate by parts:

$$u = t \qquad dv/dt = e^{-st}$$
$$du/dt = 1 \qquad v = -e^{-st}/s$$

$$\begin{aligned}
\mathcal{L}\{t\} &= \lim_{M\to\infty}\left\{\left[-\frac{te^{-st}}{s}\right]_{t=0}^{M} - \int_0^M \frac{-e^{-st}}{s}\,\mathrm{d}t\right\} \\
&= \lim_{M\to\infty}\left\{0 - \frac{Me^{-sM}}{s} + \frac{1}{s}\int_0^M e^{-st}\,\mathrm{d}t\right\} \\
&= \underbrace{\lim_{M\to\infty}\left\{-\frac{Me^{-sM}}{s}\right\}}_{\substack{0:\text{ since } s>0,\ e^{-sM}\to 0 \\ \text{faster than } M\to\infty}} + \frac{1}{s}\underbrace{\lim_{M\to\infty}\left\{\int_0^M e^{-st}\,\mathrm{d}t\right\}}_{1/s \text{ from previous result}} \\
\mathcal{L}\{t\} &= \frac{1}{s^2}
\end{aligned}$$

**Result 3** $f(t) = t^n$. In this case we will state the result and then prove it by induction.

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

First we check the **base case** $n = 1$:

$$f(t) = t \qquad F(s) = \frac{1}{s^2} \qquad \frac{1!}{s^{1+1}} = \frac{1}{s^2}$$

Now we **assume** it is true for $n \le k$ and try to prove it for $n = k+1$. If we can, then we're done.

$$f(t) = t^{k+1} \qquad F(s) = \int_0^\infty e^{-st}t^{k+1}\,\mathrm{d}t$$

Now we can use the "multiply by $-t$" rule and the "multiply by a constant rule" to say

$$\mathcal{L}\{t^{k+1}\} = \mathcal{L}\{(-t)(-t^k)\} = \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{(-t^k)\} = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{t^k\}$$

and then use our assumption (for $n = k$) to give

$$\mathcal{L}\{t^{k+1}\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{k!}{s^{k+1}}\right) = -\frac{\mathrm{d}}{\mathrm{d}s}(k!s^{-(k+1)}) = -(k![-(k+1)]s^{-(k+1)-1}) = (k+1)k!s^{-(k+2)}$$

$$\mathcal{L}\{t^{k+1}\} = \frac{(k+1)!}{s^{(k+1)+1}}$$

which is the right form for $n = k+1$ so the form is true for all $n \ge 1$.

**Result 4** $f(t) = e^{at}$ with $a$ a constant.

$$\begin{aligned}
F(s) &= \mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at}\,\mathrm{d}t \\
&= \int_0^\infty e^{-(s-a)t}\,\mathrm{d}t = \lim_{M\to\infty}\left[\frac{-1}{(s-a)}e^{-(s-a)t}\right]_{t=0}^{M} = \frac{-1}{(s-a)}(0-1) = \frac{1}{(s-a)} \\
\mathcal{L}\{e^{at}\} &= \frac{1}{(s-a)}.
\end{aligned}$$

Note in this case we need $s > a$ for the integral to converge.

We could have proved this more easily using **Rule 3** and **Result 1**:

$$\mathcal{L}\{e^{at}f(t)\} = F(s{-}a) \qquad \text{and} \qquad \mathcal{L}\{1\} = \frac{1}{s} \qquad \text{so} \qquad \mathcal{L}\{e^{at}{\times}1\} = \frac{1}{(s-a)}.$$

**Results 5 & 6** $\sin(at)$ and $\cos(at)$.

There are two ways to do these: a laborious but straightforward way, and a much easier way that needs you to remember complex numbers. First the laborious way (we'll do cos).

$$F(s) = \mathcal{L}\{\cos(at)\} = \int_0^\infty e^{-st}\cos(at)\,\mathrm{d}t$$

By parts:

$$u = \cos(at) \qquad \mathrm{d}v/\mathrm{d}t = e^{-st}$$
$$\mathrm{d}u/\mathrm{d}t = -a\sin(at) \qquad v = -e^{-st}/s$$

$$\begin{aligned} F(s) &= [-\cos(at)e^{-st}/s]_{t=0}^{\infty} - \int_0^\infty -a\sin(at)(-e^{-st}/s)\,\mathrm{d}t \\ &= \frac{1}{s} - \frac{a}{s}\int_0^\infty \sin(at)e^{-st}\,\mathrm{d}t \end{aligned}$$

By parts again:

$$u = \sin(at) \qquad \mathrm{d}v/\mathrm{d}t = e^{-st}$$
$$\mathrm{d}u/\mathrm{d}t = a\cos(at) \qquad v = -e^{-st}/s$$

$$\begin{aligned} F(s) &= \frac{1}{s} - \frac{a}{s}\left\{[\sin(at)(-e^{-st}/s)]_{t=0}^{\infty} - \int_0^\infty a\cos(at)(-e^{-st}/s)\,\mathrm{d}t\right\} \\ &= \frac{1}{s} - \frac{a}{s}\left\{\frac{a}{s}\underbrace{\int_0^\infty \cos(at)e^{-st}\,\mathrm{d}t}_{F(s)}\right\} \\ F(s) &= \frac{1}{s} - \frac{a^2}{s^2}F(s) \qquad\qquad F(s) = \mathcal{L}\{\cos(at)\} = \frac{s}{(s^2+a^2)}. \end{aligned}$$

In the same way, we can derive

$$\mathcal{L}\{\sin(at)\} = \frac{a}{(s^2+a^2)}.$$

However, if we are prepared to use complex numbers we can get **both at once!**. Recall the key facts:

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad \cos\theta = \Re(e^{i\theta}) \qquad \sin\theta = \Im(e^{i\theta}).$$

It follows (if we use our rules for adding functions and multiplying by a constant, and the observation that the Laplace transform of a real function is real) that

$$\mathcal{L}\{\cos(at)\} = \Re\mathcal{L}\{e^{iat}\} \qquad \text{and} \qquad \mathcal{L}\{\sin(at)\} = \Im\mathcal{L}\{e^{iat}\}$$

We already know that

$$\mathcal{L}\{e^{at}\} = \frac{1}{(s-a)} \qquad \text{so} \qquad \mathcal{L}\{e^{iat}\} = \frac{1}{(s-ia)}.$$

Now we just need to manipulate this to get real and imaginary parts:

$$\begin{aligned}
\mathcal{L}\{\cos(at)\} + i\mathcal{L}\{\sin(at)\} &= \mathcal{L}\{e^{iat}\} = \frac{1}{(s-ia)} = \frac{(s+ia)}{(s-ia)(s+ia)} \\
&= \frac{(s+ia)}{(s^2+a^2)} = \frac{s}{(s^2+a^2)} + i\frac{a}{(s^2+a^2)}
\end{aligned}$$

$$\mathcal{L}\{\cos(at)\} = \frac{s}{(s^2+a^2)} \qquad \mathcal{L}\{\sin(at)\} = \frac{a}{(s^2+a^2)}$$

**Results 7 & 8** $\sinh(at)$ and $\cosh(at)$

Here we just work straight from the definitions:

$$\sinh(at) = \frac{e^{at} - e^{-at}}{2}$$

$$\begin{aligned}
\mathcal{L}\{\sinh(at)\} &= \mathcal{L}\left\{\frac{e^{at} - e^{-at}}{2}\right\} = \frac{1}{2}\mathcal{L}\{e^{at}\} - \frac{1}{2}\mathcal{L}\{e^{-at}\} = \frac{1}{2}\frac{1}{(s-a)} - \frac{1}{2}\frac{1}{(s+a)} \\
&= \frac{1}{2}\left\{\frac{(s+a)}{(s-a)(s+a)} - \frac{(s-a)}{(s-a)(s+a)}\right\} = \frac{a}{(s^2-a^2)}
\end{aligned}$$

In the same way,

$$\mathcal{L}\{\cosh(at)\} = \frac{s}{(s^2-a^2)}.$$

### 0.0.6 More complicated functions

To create the Laplace transform of functions we haven't seen before, we can either work directly from the definition and integrate, or use the rules and results we've already seen. Here are a few examples.

1. $f(t) = t^2$. We can use our rule for $t^n$ with $n = 2$:

$$\mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}.$$

2. $f(t) = te^{-3t}$. We start with our rule for multiplying by $t$:

$$\mathcal{L}\{te^{-3t}\} = -\mathcal{L}\{(-t)e^{-3t}\} = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{e^{-3t}\} = -\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{s+3}\right) = -\left(\frac{-1}{(s+3)^2}\right) = \frac{1}{(s+3)^2}.$$

3. $f(t) = e^{at}\cos\omega t$, $a < 0$.

   Because we have a rule for multiplying by $e^{at}$ we can do this in two steps, starting with the cos term:

$$\mathcal{L}\{\cos\omega t\} = \frac{s}{(s^2+\omega^2)} = F_1(s)$$

Then

$$\mathcal{L}\{e^{at}\cos\omega t\} = F_1(s-a) = \frac{(s-a)}{((s-a)^2+\omega^2)}.$$

On the other hand, we could get the cos and sin versions of this result both at once by a single integration if we're not afraid of complex numbers:

$$\begin{aligned}
\mathcal{L}\{e^{(a+i\omega)t}\} &= \int_0^\infty e^{-st}e^{(a+i\omega)t}\,\mathrm{d}t = \int_0^\infty e^{(a-s+i\omega)t}\,\mathrm{d}t \\
&= \left[\frac{e^{(a-s+i\omega)t}}{(a+s+i\omega)}\right]_{t=0}^{\infty} \quad \text{provided } a-s \text{ and } \omega \text{ are not both } 0 \\
&= \frac{\lim_{M\to\infty} e^{(a-s+i\omega)M}-1}{(a-s+i\omega)} = \frac{-1}{(a-s+i\omega)} \quad \text{provided } s>a \\
&= \frac{1}{(s-a-i\omega)} = \frac{s-a+i\omega}{((s-a)^2+\omega^2)}
\end{aligned}$$

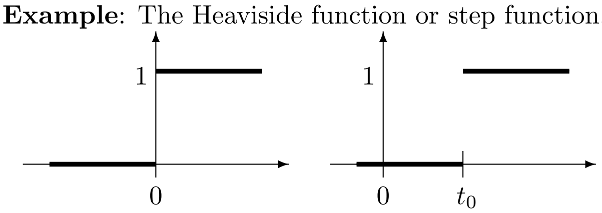
So, equating real and imaginary parts,

$$\mathcal{L}\{e^{at}\cos\omega t\} = \frac{s-a}{((s-a)^2+\omega^2)} \qquad \mathcal{L}\{e^{at}\sin\omega t\} = \frac{\omega}{((s-a)^2+\omega^2)}.$$

### 0.0.7 Discontinuous function example

To transform a discontinuous function, we just carry out the integration over the different ranges separately.

**Example**: The Heaviside function or step function.




$$H(t) = \begin{cases} 0 & t<0 \\ 1 & t\geq 0 \end{cases} \qquad H(t-t_0) = \begin{cases} 0 & t<t_0 \\ 1 & t\geq t_0 \end{cases}$$

To find the Laplace transform:

$$\begin{aligned}
\mathcal{L}\{H(t-t_0)\} &= \int_0^\infty e^{-st}H(t-t_0)\,\mathrm{d}t = \int_0^{t_0} e^{-st}0\,\mathrm{d}t + \int_{t_0}^\infty e^{-st}1\,\mathrm{d}t \\
&= \int_{t_0}^\infty e^{-st}\,\mathrm{d}t = \lim_{M\to\infty}\left[\frac{-e^{-st}}{s}\right]_{t=t_0}^{M} = \lim_{M\to\infty}\left\{\frac{e^{-st_0}}{s} - \frac{e^{-sM}}{s}\right\} = \frac{e^{-t_0 s}}{s}
\end{aligned}$$

providing $s>0$.

### 0.0.8 Inverting the Laplace transform

There is a systematic method of inverting Laplace transforms: but it involves integration in the complex plane and we won't cover it in this course.

Instead, we will use a combination of **look-up tables** and **spotting a pattern** to invert the functions we come across. Essentially, if you are given $F(s)$ and you want to find the function it came from, try various likely functions $f(t)$ and when you find one that works, you're there.

**Examples**

1.
$$F(s) = \frac{4}{s^2+16} = \frac{4}{s^2+4^2}$$

This matches exactly one of our standard forms: so

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{s^2+16}\right\} = \sin(4t)$$

2.
$$F(s) = \frac{30}{s^2+9} = 10 \times \frac{3}{s^2+3^2}$$

Now we have a multiple of a standard form:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{30}{s^2+9}\right\} = 10\sin(3t)$$

3.
$$F(s) = \frac{2}{s^2} = 2 \times \frac{1}{s^2} = 2 \times \frac{1!}{s^{1+1}}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2}\right\} = 2t^1 = 2t.$$

4.
$$F(s) = \frac{1}{(s-3)^2}$$

We can see that this is related to $1/s^2$, so let's look at that first:

$$F_1(s) = \frac{1}{s^2} \qquad f_1(t) = t$$

$$F(s) = F_1(s-3) \qquad f(t) = e^{3t} f_1(t) = te^{3t}.$$

**Recap**

We are looking at ways to find the inverse Laplace transform of complicated functions, using just our rules 1-7 and results 1-8 (on handout 17).

Here is the final example:

5.
$$F(s) = \frac{s+5}{s^2+8s+24}$$

In this case we need to rearrange and spot patterns:

$$\begin{aligned}
F(s) &= \frac{s+5}{(s+4)^2-16+24} = \frac{s+5}{(s+4)^2+8} = \frac{(s+4)+1}{(s+4)^2+(2\sqrt{2})^2} \\
&= \frac{s+4}{(s+4)^2+(2\sqrt{2})^2} + \frac{1}{(s+4)^2+(2\sqrt{2})^2} \\
&= \frac{s+4}{(s+4)^2+(2\sqrt{2})^2} + \frac{1}{2\sqrt{2}}\frac{2\sqrt{2}}{(s+4)^2+(2\sqrt{2})^2}
\end{aligned}$$

Now let $z = s + 4$ and $a = 2\sqrt{2}$:

$$F(s) = \frac{z}{z^2 + a^2} + \frac{1}{2\sqrt{2}} \frac{a}{z^2 + a^2}$$

Now these two look familiar: consider two known results:

$$f_1(t) = \cos(at) \qquad F_1(s) = \frac{s}{s^2 + a^2}$$

$$f_2(t) = \sin(at) \qquad F_2(s) = \frac{a}{s^2 + a^2}$$

We can combine them:

$$F(s) = F_1(z) + \frac{1}{2\sqrt{2}} F_2(z) = F_1(s+4) + \frac{1}{2\sqrt{2}} F_2(s+4)$$

Now we can use the adding-functions rule:

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\{F_1(s+4)\} + \frac{1}{2\sqrt{2}} \mathcal{L}^{-1}\{F_2(s+4)\} \\
&= e^{-4t} f_1(t) + e^{-4t} \frac{1}{2\sqrt{2}} f_2(t) \\
&= e^{-4t} \cos(2\sqrt{2}\, t) + e^{-4t} \frac{1}{2\sqrt{2}} \sin(2\sqrt{2}\, t).
\end{aligned}$$

### 0.0.9 Applications

Although you already know how to solve a linear ODE with constant coefficients, we'll look at a couple of examples to see how Laplace transform methods work with them; then we'll move onto something you maybe can't solve already.

**Example**

$$\frac{\mathrm{d}f}{\mathrm{d}t} = -\lambda f(t) \qquad f(0) = f_0$$

Taking Laplace transforms:

$$\mathcal{L}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}\right\} = \mathcal{L}\{-\lambda f(t)\} = -\lambda \mathcal{L}\{f(t)\}$$

$$sF(s) - f(0) = -\lambda F(s)$$

$$(s + \lambda) F(s) = f(0) \qquad F(s) = \frac{f_0}{(s + \lambda)}$$

So we had a simple algebraic system to solve for $F(s)$; now we just need to invert to get the solution for $f(t)$:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{f_0}{(s+\lambda)}\right\} = f_0 \mathcal{L}^{-1}\left\{\frac{1}{(s+\lambda)}\right\} = f_0 e^{-\lambda t}.$$

This particular example arises in, for example, radioactive decay, where the rate of loss of material is proportional to the amount of material left.

**Example**

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} - \frac{\mathrm{d}f}{\mathrm{d}t} - 6f = 0 \qquad f(0) = 2,\ \frac{\mathrm{d}f}{\mathrm{d}t}(0) = -1.$$

We take the Laplace transform of the whole equation:

$$\mathcal{L}\left\{\frac{\mathrm{d}^2 f}{\mathrm{d}t^2}\right\} - \mathcal{L}\left\{\frac{\mathrm{d}f}{\mathrm{d}t}\right\} - 6\mathcal{L}\{f\} = 0$$

$$s^2F(s) - sf(0) - f'(0) - \{sF(s) - f(0)\} - 6F(s) = 0$$

Now we put in the parts we know: $f(0)$ and $f'(0)$:

$$s^2F(s) - 2s + 1 - \{sF(s) - 2\} - 6F(s) = 0$$

$$(s^2 - s - 6)F(s) = 2s - 3 \qquad F(s) = \frac{2s-3}{(s^2 - s - 6)}$$

In order to get this in a form we can invert, we use **partial fractions** (remember E001 last year!):

$$\frac{2s-3}{(s^2-s-6)} = \frac{2s-3}{(s-3)(s+2)} = \frac{A}{(s-3)} + \frac{B}{(s+2)}$$

Multiplying out gives:

$$2s - 3 = A(s+2) + B(s-3)$$

so at $s = 3$ we see $3 = 5A$ and at $s = -2$, $-7 = -5B$ so

$$F(s) = \frac{3}{5(s-3)} + \frac{7}{5(s+2)} \qquad \text{and} \qquad f(t) = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}.$$

### 0.0.10 Systems of linear ODEs

If we have more than one ODE at once, like ordinary simultaneous equations but with derivatives as well, then the standard methods you learnt last year can't be used; but Laplace transforms save the day.

**Example**

$$y'' + z + y = 0 \qquad z' + y' = 0$$
$$y_0 = 0 \quad y_0' = 0 \quad z_0 = 1$$

We take the Laplace transform of both equations: let the transform of $y(t)$ be $Y(s)$ and the transform of $z(t)$ be $Z(s)$.

$$s^2Y(s) - sy_0 - y_0' + Z(s) + Y(s) = 0 \qquad sZ(s) - z_0 + sY(s) - y_0 = 0$$

$$s^2Y(s) - 0 - 0 + Z(s) + Y(s) = 0 \qquad sZ(s) - 1 + sY(s) - 0 = 0$$

This is now a pair of simultaneous equations for $Y$ and $Z$:

$$\begin{array}{cccc} (s^2+1)Y(s) & + & Z(s) & = & 0 \\ sY(s) & + & sZ(s) & = & 1 \end{array} \qquad \begin{pmatrix} (s^2+1) & 1 \\ s & s \end{pmatrix} \begin{pmatrix} Y(s) \\ Z(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and we could solve it like all the matrix systems we did at the beginning of term, or just try to eliminate a variable the easiest way possible.

$$r_2 - sr_1 : \qquad [s - s(s^2+1)]Y(s) = 1 \qquad Y(s) = -\frac{1}{s^3}.$$

Substituting back into $r_2$:

$$-\frac{1}{s^2} + sZ(s) = 1 \qquad Z(s) = \frac{1}{s} + \frac{1}{s^3}.$$

Inverting:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{-\frac{1}{s^3}\right\} = -\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = -\frac{1}{2}t^2 \\ z(t) &= \mathcal{L}^{-1}\{Z(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = 1 + \frac{1}{2}t^2 \end{aligned}$$

**Example**

$$\begin{aligned} z'' + y' &= \cos t \qquad & z_0 &= -1 \quad y_0 = 1 \\ y'' - z &= \sin t \qquad & z'_0 &= -1 \quad y'_0 = 0 \end{aligned}$$

Again, we transform the differential equations:

$$\mathcal{L}\{z''\} + \mathcal{L}\{y'\} = \mathcal{L}\{\cos t\} \qquad \mathcal{L}\{y''\} - \mathcal{L}\{z\} = \mathcal{L}\{\sin t\}$$

$$s^2Z(s) - sz_0 - z'_0 + sY(s) - y_0 = \frac{s}{(s^2+1)} \qquad s^2Y(s) - sy_0 - y'_0 - Z(s) = \frac{1}{(s^2+1)}$$

$$s^2Z(s) + s + 1 + sY(s) - 1 = \frac{s}{(s^2+1)} \qquad s^2Y(s) - s - Z(s) = \frac{1}{(s^2+1)}$$

$$\begin{pmatrix} s & s^2 \\ s^2 & -1 \end{pmatrix} \begin{pmatrix} Y(s) \\ Z(s) \end{pmatrix} = \begin{pmatrix} s/(s^2+1) - s \\ 1/(s^2+1) + s \end{pmatrix} = \begin{pmatrix} -s^3/(s^2+1) \\ (s^3+s+1)/(s^2+1) \end{pmatrix}$$

$sr_1 - r_2$:

$$(s^3+1)Z(s) = s\left(\frac{-s^3}{(s^2+1)}\right) - \frac{(s^3+s+1)}{(s^2+1)} = \frac{-s^4 - s^3 - s - 1}{(s^2+1)} = \frac{-(s+1)(s^3+1)}{(s^2+1)}$$

so

$$Z(s) = -\frac{(s+1)}{(s^2+1)} = -\frac{s}{(s^2+1)} - \frac{1}{(s^2+1)}$$

Substituting back into $r_1$:

$$sY(s) - \frac{s^2(s+1)}{(s^2+1)} = -\frac{s^3}{(s^2+1)} \qquad Y(s) = \frac{s}{(s^2+1)}$$

Inverting back, these are easily related to standard forms:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)}\right\} = \cos t \\ z(t) &= \mathcal{L}^{-1}\{Z(s)\} = \mathcal{L}^{-1}\left\{-\frac{s}{(s^2+1)} - \frac{1}{(s^2+1)}\right\} = -\cos t - \sin t. \end{aligned}$$