SPA6311 Physical Cosmology 2015

Lectures 5.1, 5.2 and 5.3

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Recall

$$H^2 = H_0^2\left(\Omega_{m_0}\,(1+z)^3 + \Omega_{r_0}\,(1+z)^4 + \Omega_\Lambda\right) - \frac{kc^2}{a^2}$$

Define curvature density

$$\Omega_{k_0} = \frac{-kc^2}{H_0^2}$$

get:

$$-\frac{kc^2}{a^2} = -kc^2(1+z)^3 = \Omega_{k_0}(1+z)^2 H_0^2$$

and Friedmann

$$H^2 = H_0^2\left(\Omega_{m_0}\,(1+z)^3 + \Omega_{r_0}\,(1+z)^4 + \Omega_{k_0}(1+z)^2 + \Omega_\Lambda\right)$$

Can be solved numerically.

The deceleration parameter q

Consider, e.g. matter dominated universe

$$a(t) \propto t^{\frac{2}{3}}$$

Note: expansion slows down. This can be made more precise, by Taylor expanding the scale factor:

$$a(t) = a(t_0) + \dot{a}(t_0)(t-t_0) + \frac{1}{2}\ddot{a}(t_0)(t-t_0)^2 + ...$$

Rewrite

$$a(t) = a(t_0)\left(1 + H_0(t-t_0) - \frac{1}{2}q_0 H_0^2(t-t_0)^2\right)$$

where we define

$$q_0 = -\frac{\ddot{a}(t_0)}{a(t_0)}\frac{1}{H_0^2}$$

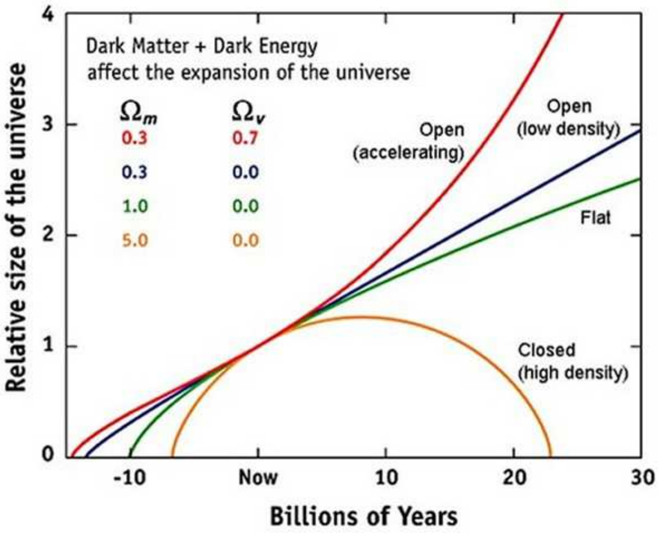

Figure 1: The evolution of the scale factor in different universes, $\Lambda = 0$.

deceleration parameter. Or

$$q_0 = \frac{-\ddot{a}(t_0)a(t_0)}{(\dot{a}(t_0))^2}$$

Recall acceleration equation

$$\begin{aligned} \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3P) \\ q_0 &= \frac{4\pi G}{3}(\rho_0 + 3P_0)\frac{1}{H_0} \\ q_0 &= (1+3\omega)\frac{\Omega_0}{2} \end{aligned}$$

Knowing q_0, could get us Ω_0, if all matter us described by $(1+\omega)\,\rho$ and $k = 0$.

Note: q_0 named deceleration parameter and defined with minus sign in definition for historical reasons: It was assumed that $\frac{\ddot{a}}{a} < 0$ or $(\rho + 3P) > 0$. However, observations today point $\frac{\ddot{a}}{a} > 0$ and $q_0 < 0$

Recall

The FRW line element

$$ds^2 = -c^2dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega_{(2)}^2\right)$$

$a(t)$ is homogeneous and isotropic. Light travels on null geodesics, that is $ds = 0$. Hence for a light ray travelling radially outwards, $d\Omega_{(2)} = 0$, we have

$$\frac{c\ dt}{a} = \frac{dr}{\sqrt{1-kr^2}}$$

Distances in FRW

1. **Proper distance**

   the length of the spatial geodesic at some fixed time t.

$$d_p = \int ds$$

The proper distance from us (at point (0,0,0)) to a galaxy situated (r_0,0,0) is

$$\begin{aligned} d_p(t) &= \int_0^{r_0} ds \\ &= a(t)\int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}} \end{aligned}$$

Today, with $a(t_0) = 1$

$$d_p = \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}}$$

In a flat universe

$$d_p = \int_0^{r_0} dr = r_0$$

For a closed universe, $k > 0$ use $U = \sqrt{k}r$ and $du = \sqrt{k}dr$, get

$$d_p = \frac{1}{\sqrt{k}}\int_0^{\sqrt{k}r_0} \frac{dU}{\sqrt{1-U^2}}$$

Using table

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin{(x)}$$

So

$$d_p = \frac{1}{\sqrt{k}} \arcsin\left(\sqrt{k} r_0\right)$$

Similarly, for an open universe

$$d_p = \frac{1}{\sqrt{k}} \text{arcsinh}\left(\sqrt{|k|} r_0\right)$$

2. **Luminosity distance**

$$l = \frac{L}{4\pi D^2}$$

This assumed Euclidean geometry: a flat non-expanding universe. L is the energy emitted per second per solid angle.

Total light output of the source is $4\pi L$. By our observer light is spread out over a sphere of area $4\pi d_{lum}^2$. The receiver receives $\frac{1}{4\pi d_{lum}^2}$ of the total output.

Define flux density

$$S = \frac{4\pi L}{4\pi d_{lum}^2} = \frac{L}{d_{lum}^2}$$

Inverted

$$d_{lum}^2 = \frac{L}{S}$$

Take into account two effects:

- Redshift of photon energy $E = \hbar\omega$, $E \propto \omega \propto a^{-1} \propto (1+z)$
- Photons reach observer less frequently: photons emitted at intervals Δt_{em}, will be received at intervals

$$\Delta t_{em} \frac{a(t_0)}{a(t_{em})} = \Delta t_{em}(1+z)$$

This gives for an object at r_0

$$S = \frac{L}{r_0^2 (1+z)^2}$$

and hence

$$d_{lum} = (1+z) r_0$$

that is distant objects appear further away then they actually are, as redshift reduces their apparent luminosity.

Recall r_0 depends on the geometry of the universe. $k = 0$; $d_p = r_0 \rightarrow d_{lum} = d_p(1+z)$. For $z << 1$, $d_{lum} \approx d_p$

$k > 0$

$$r_0 = \frac{1}{\sqrt{k}} \sin\left(\sqrt{k} d_p\right)$$

which implies

$$d_{lum} = \frac{1+z}{\sqrt{k}} \sin\left(\sqrt{k} d_p\right)$$

For $z << 1$ get

$$\begin{aligned} d_{lum} &\approx \frac{1}{\sqrt{k}}\left(\sqrt{k} d_p + ...\right) \\ &\approx d_p \end{aligned}$$

and similarly for the open case.

3. **Angular diameter distance**: This distance, denoted d_{diam}, is defined: the distance an object of known physical extent appears to be at, assuming Euclidean geometry (ie it is a measure of how large objects appear). Object of size l, perpendicular to the line of sight then

$$d_{diam} = \frac{l}{tan\,(\theta)} \approx \frac{l}{\theta}$$

so

$$l = ds = r_0 a(t_{em}) d\theta$$

$\rightarrow$

$$d\theta = \frac{l}{r_0 a(t_{em})}$$

Using now $a(t_{em}) = \frac{1}{1+z}$

$$d\theta = \frac{l}{r_0}(1+z)$$

and hence

$$d_{diam} = \frac{l}{\frac{l}{r_0}(1+z)} = \frac{r_0}{1+z}$$

For nearby objects

$$d_{diam} \approx r_0$$

E.g: $k = 0$, matter dom:

$$\begin{aligned} r_{em} &= \int_{t_{em}}^{t_0} \frac{cdt}{a} \\ &= ct_0^{\frac{2}{3}} \int_{t_{em}}^{t_0} \frac{dt}{t^{\frac{2}{3}}} \\ &= 3ct_0 \left(1 - \left(\frac{t_{em}}{t_0}\right)^{\frac{1}{3}}\right) \\ &= 3ct_0 \left(1 - a_{em}^{\frac{1}{2}}\right) \\ &= 3ct_0 \left(1 - \frac{1}{\sqrt{1+z}}\right) \end{aligned}$$

Now, for an object with extent l

$$\begin{aligned} \theta &= \frac{l(1+z)}{r_{em}} \\ &= \frac{l}{3ct_0} \frac{(1+z)^{\frac{3}{2}}}{\sqrt{1+z} - 1} \end{aligned}$$

For $z << 1$: $\theta \propto \frac{1}{z}$ and for $z >> 1$: $\theta \propto z$. d_{diam} decreases at large $z \rightarrow$ distant objects appear larger.

Note: it is not straight forward to get the information we want (eg distance D) from the observations (z, l)

Horizons (in FRW)

In general: roughly speaking, a horizon is the boundary between regions we can observe (exchange information), and ones we can't.

The Cosmological Horizon: The maximum distance light has travelled since the Big Bang (at $t = 0$). Denoted r_H, it is given by the following equation

$$\int_0^t \frac{c\ dt}{a} = \int_0^{r_H} \frac{dr}{\sqrt{1 - kr^2}}$$

For $k = 0$ this is

$$c \int_0^t \frac{dt}{a} = \int_0^{r_H} dr$$

Which gives us

$$r_H = c\int_0^t \frac{dt}{a}$$

To get physical distance, multiply comoving distance by $a(t)$

$$R_H = ar_H = a(t)c\int_0^t \frac{dt}{a}$$

If we choose $a(t_0) = 1$ today, we have $R_H(t_0) = r_H(t_0)$

For matter dom ($k = 0$, $\Lambda = 0$)

$$a = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}$$

$$\int_0^t \frac{dt'}{a(t')} = 3t_0^{\frac{2}{3}}t^{\frac{1}{3}}$$

So

$$R_H(t_0) = c3t_0$$

This is the maximum distance light can have travelled (during matter domination). Note: stars (and galaxies) formed during matter domination, hence $3ct_0$ is the furthest distance starlight can have travelled. This resolves Olbers' paradox. Note, without the expansion of the universe the Cosmological Horizon would just be ct_0

Sometimes the cosmological horizon is referred to as particle horizon have speed $v \leq c$, it is an upper limit particles could have travelled since $t = 0$

The Event Horizon: is denoted by r_{ev} and is defined as the comoving radius within which signals emitted at time t can be observed by the time t_{max}:

$$\int_t^{t_{max}} \frac{cdt'}{a(t')} = \int_0^{r_{ev}} \frac{dr}{\sqrt{1-kr^2}}$$

For $k = 0$

$$r_{ev} = c\int_t^{t_{max}} \frac{dt'}{a(t')}$$

The physical radius:

$$R_{ev}(t) = a(t)r_{ev}(t)$$

For $\Lambda = 0$:

- In the open case ($k < 0$) t_{max} is unbounded and $R_{ev} \to \infty$
- In the close case ($k > 0$), t_{max} is finite, and so is R_{ev}.

For $\Lambda \neq 0$ we found (Λ domination)

$$a \propto e^{\left(\sqrt{\frac{\Lambda}{3}}ct\right)}$$

In this case, get for $R_{ev}(t)$

$$\beta = \sqrt{\frac{\Lambda}{3}}$$

So

$$\begin{aligned} R_{ev}(t) &= a(t)r_{ev}(t) \\ &= ce^{\beta t}\int_t^{t_{max}} dt' e^{-\beta t'} \end{aligned}$$

So

$$R_{ev}(t) = \frac{c}{\beta}\left(1 - e^{\beta(t-t_{max})}\right)$$

as $t_{max} \to \infty$, $e^{\beta(t-t_{max})} \to 0$. Hence, in this case R_{ev} is finite (indeed $R_{ev} = \sqrt{\frac{\Lambda}{3}}$). There is a limit to what we can "see" in the future.

The age of the universe

Rough estimate: Assume constant expansion rate $\dot{a}(t) = \text{const}$. Today $H_0 = \frac{\dot{a}(t_0)}{a(t_0)} \to \dot{a}(t_0) = H_0$ if $a(t_0) = 1$. Integrate, and get

$$\begin{aligned} a(t) &= H_0 t + \text{const} \\ &= H_0 t \end{aligned}$$

Evaluate today

$$\begin{aligned} a_0 &= H_0 t_0 \\ &= 1 \end{aligned}$$

So

$$t_0 = H_0^{-1}$$

This give

$$t_0 \approx \frac{1}{100hkms^{-1}Mpc^{-1}} \approx 10Gyr$$

Unfortunately, expansion rate is not constant. It depends on the matter content and the geometry of the universe (Friedmann Equation).

E.g assume matter domination, and get $a(t) \propto t^{\frac{2}{3}}$

$$H = \frac{\dot{a}}{a} = \frac{2}{3}t^{-1}$$

Evaluate at today

$$H_0 = \frac{2}{3}t_0^{-1}$$

Or

$$t_0 = \frac{2}{3}H_0^{-1} \approx 6.6Gyr$$

Observational limits on the age

The universe must be older than the objects it contains. Some age measurements:

- Geological data Earth roughly 5 Gyr old
- Uranium Isotopes (produced in supernovae) indicate through decay chains and rates that Milky Way is roughly 7 Gyr old
- Studies of old white dwarfs and globular clusters indicate both are roughly 10 Gyr. Adding roughly 1Gyr for globular clusters and white dwarfs to form; got only very rough agreement with original estimate

Accurate Calculation

$$\begin{aligned} t_0 &= \int_0^{t_0} dt \\ &= \int_0^{a_0} \frac{da}{\dot{a}} \\ &= \int_0^{a_0} \frac{da}{aH} \end{aligned}$$

Recall def of redshift

$$1 + z = \frac{a_0}{a}$$

Differentiating this

$$dz = -\frac{a_0}{a^2}da$$

Or writing it more nicely

$$\frac{dz}{1+z} = \frac{-da}{a}$$

get

$$\begin{aligned} t_0 &= -\int_{\infty}^{0} \frac{dz}{(1+z)H(z)} \\ &= \int_{0}^{\infty} \frac{dz}{(1+z)H(z)} \end{aligned}$$

Friedmann equation from beginning of lecture. Can solve numerically, and find

$$t_0 = 13.8\ Gyr = 4.3 \times 10^{17}\ s$$

According to Planck satellite data from 2013.