Week 9

1 Evolution after the main sequence

As a star evolves through the burning of hydrogen to helium in its core, the mean molecular weight, μ, increases and this has the effect of decreasing the pressure deep in the core (which for an ideal gas is $P = \frac{\mathcal{R}}{\mu}\rho T$, where the gas constant can be expressed as $\mathcal{R} = k_{\rm B}/m_{\rm H}$). Gravitational contraction of the core causes it to heat up, and this in turn leads to enhanced energy generation such that stars tend to increase their luminosity somewhat during their main sequence life times. Towards the end of main sequence life times, when nuclear fuels deep in stellar interiors become exhausted, significant contraction of stellar cores can arise, and the resulting increases in density leads to the ideal gas approximation breaking down as quantum mechanical and relativistic effects become important.

1.1 The equation of state of a degenerate gas

As the density of matter increases deep in stellar interiors during the late stages of stellar evolution, we need to take into account quantum mechanical effects that change the equation of state. Of particular importance is the Pauli exclusion principle which states: *no more than two electrons (of opposite spin) can occupy the same quantum state.*

The quantum state of an electron is given by the 6 values which describe its position and momentum (i.e. x, y, z, p_x, p_y, p_z). It is not possible, however, to know both the position and momentum of an electron with complete accuracy. There is a fundamental uncertainty δr in any position coordinate and δp in any corresponding momentum coordinate, such that

$$\delta r\, \delta p \geq \frac{h}{4\pi}. \tag{1}$$

This is known as Heisenberg's uncertainty principle and it means that, instead of thinking of a quantum state as a point in six-dimensional position-momentum space (also known as *phase* space), we can think of a quantum state as a volume, h^3, of phase space. Pauli's exclusion principle then tells us that no more that 2 electrons are permitted to occupy each volume h^3 of phase space.

Now consider what happens at the centre of a star as the density of electrons is increased. The electrons will become increasingly crowded together in position space, and eventually a situation will be reached when two electrons with almost the same momenta will occupy the same point in space. This volume of phase space will then be full (according to Pauli's exclusion principle). It will not be possible to squeeze in an additional electron to this region of position space unless its momentum is significantly different (so that it occupies a different region of phase space). The additional electron will therefore possess a higher momentum than it would have had at the same temperature in an ideal gas. Higher momentum implies that such a gas will exert a greater pressure than predicted by the ideal gas equation of state.

A gas in which the Pauli exclusion principle is important is known as a *degenerate* gas. Because the ions in such a gas have higher momenta than the electrons, they are less likely to violate Pauli's exclusion principle (another way of thinking about this is to say that the ions have smaller de Broglie wavelengths than electrons, and degeneracy effects start to occur when the de Broglie wavelengths of particles start to overlap). The pressure due to the ions can then be treated as an ideal gas, but the pressure due to the degenerate electrons is much larger and hence the gas obeys a different equation of state, which we shall now derive.

Consider a group of electrons occupying a volume V of position space which have momenta lying in the range p and $p + \delta p$. The volume of momentum space occupied by these electrons is given by the volume of a spherical shell of radius p and thickness δp:

$$4\pi p^2 \delta p.$$

The volume of phase space occupied by these electrons is given by the volume they occupy in position space multiplied by the volume they occupy in momentum space:

$$4\pi p^2 V \delta p.$$

The number of quantum states in this volume of phase space is then simply the above volume divided by the volume of a quantum state, h^3:

$$\left(\frac{4\pi p^2 V}{h^3}\right)\delta p.$$

If we now define $N_p \delta p$ as the number of electrons in V with momenta in the range p and $p+\delta p$ (so N_p is the number of electrons per unit momentum range in volume V), Pauli's exclusion principle tells us that

$$N_p \delta p \leq \left(\frac{8\pi p^2 V}{h^3}\right)\delta p. \tag{2}$$

i.e. The number of states with at most 2 electrons occupying these states.

We will now define a *completely degenerate gas* as one in which all of the momentum states up to some critical value, p_0, are filled, while the states with momenta p_0 are empty. Strictly speaking this occurs at absolute zero. Hence, we have

$$\begin{aligned} N_p &= \frac{8\pi p^2 V}{h^3} \quad \text{for } p \leq p_0 \\ N_p &= 0 \qquad\quad \text{for } p > p_0. \end{aligned} \tag{3}$$

The total number of electrons, N, in volume V is then given by

$$N = \frac{8\pi V}{h^3}\int_0^{p_0} p^2 dp = \frac{8\pi p_0^3 V}{3h^3}. \tag{4}$$

The pressure, P, of a gas is the mean rate of transport of momentum across unit area. This can be written as follows:

$$\begin{aligned} P &= \frac{1}{3}\int_0^\infty \frac{N_p}{V}\, p\, v_p\, dp \\ &= \frac{1}{3}\int_0^\infty n(p)\, p\, v_p\, dp. \end{aligned} \tag{5}$$

where v_p is the velocity of an electron with momentum p, and $n(p) = N_p/V$ is the number density of particles with momentum p.

Equation (5) may be derived as follows. Consider a gas of particles contained in a rectangular box (see figure 1). When a single particle hits the wall of the box labelled with area $S = bc$, it changes its momentum by an amount $2p_x$. The time interval between two consecutive hits is $2a/v_x$, and hence the average force on area S, produced by a single particle is $2p_x/(2a/v_x) = p_x v_x/a$. The average pressure is $p_x v_x/(a \times S) = v_x p_x/(abc) = v_x p_x/V$, where V is the box volume. Energy partition between the three degrees of freedom gives the average value $\langle v_x p_x \rangle = vp/3$, and hence the pressure produced by a single particle is $vp/(3V)$. If we have N particles with momentum p, the pressure is $vpN/(3V) = vpn/3$, where $n = N/V$ is the number density of particles with momentum p. When we have many particles described by a continuous distribution in momentum p, we replace n by $n(p)dp$ and integrate to obtain

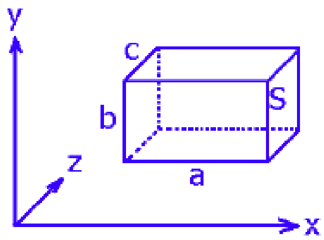

Figure 1:

$$P = \frac{1}{3}\int_0^\infty v\, p\, n(p)\, dp.$$

To evaluate the integral in eqn. (5), we must use the relation between p and v_p given by the theory of special relativity

$$p = \frac{m_e v_p}{\left(1 - \frac{v_p^2}{c^2}\right)^{1/2}},$$

which can be arranged to give

$$v_p = \frac{p}{m_e}\left(1 + \frac{p^2}{m_e^2 c^2}\right)^{-1/2},$$

where m_e is the rest mass of an electron. Hence, by combining the three expressions for N_p, P and v_p that we have obtained above, we obtain an expression for the pressure of a completely degenerate gas:

$$P = \frac{8\pi}{3h^3 m_e} \int_0^{p_0} \frac{p^4 \, dp}{\left(1 + \frac{p^2}{m_e^2 c^2}\right)^{1/2}}. \tag{6}$$

Instead of evaluating this integral in its present form, we will consider two limiting cases. First, we consider a *non-relativistic degenerate gas* (i.e. $p_0 \ll m_e c$). In this case

$$\left(1 + \frac{p^2}{m_e^2 c^2}\right)^{1/2} \to 1,$$

and hence

$$P = \frac{8\pi}{3h^3 m_e} \int_0^{p_0} p^4 \, dp = \frac{8\pi p_0^5}{15 h^3 m_e}. \tag{7}$$

Recalling from eqn. (4) that the total number of electrons in volume V

$$N = \frac{8\pi p_0^3 V}{3h^3},$$

and defining the electron number density $n_e = N/V$, we obtain

$$p_0 = \frac{h}{2}\left(\frac{3n_e}{\pi}\right)^{1/3}, \tag{8}$$

where p_0 is normally referred to as the Fermi momentum. Combining this with eqn. (7) gives

$$P = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{h^2}{m_e}\right) n_e^{5/3}. \tag{9}$$

We now consider a *ultra-relativistic degenerate gas* (i.e. $p_0 \gg m_e c$). This occurs when the velocity of an electron approaches that of light, in which case its momentum approaches infinity. In this case we can neglect 1 in comparison with $p^2/(m_e c^2)$ in the denominator of eqn. (6), i.e.

$$\left(1 + \frac{p^2}{m_e^2 c^2}\right)^{1/2} \to \frac{p}{m_e c}.$$

Hence eqn. (6) becomes

$$\frac{8\pi}{3h^3 m_e} \int_0^{p_0} m_e \, c \, p^3 \, dp = \frac{8\pi c p_0^4}{12 h^3}. \tag{10}$$

Recalling that

$$p_0 = \frac{h}{2}\left(\frac{3n_e}{\pi}\right)^{1/3},$$

and substituting into eqn. (10), we obtain

$$P = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hc n_e^{4/3}. \tag{11}$$

Our aim is to obtain an equation of state for a degenerate gas in terms of density and chemical composition so that it is compatible with the other equations of stellar structure. Hence, we must convert the electron density, n_e, to mass density, ρ. We can do this using similar arguments to those given in the derivation of the *mean molecular weight*, μ. For each hydrogen atom of mass $m_{\rm H}$ there is one electron. Hence, if the mass fraction of hydrogen is denoted by X, then the number density of electrons arising from ionised hydrogen atoms is

$$\frac{\rho}{m_{\rm H}} X.$$

Similarly, each ionised helium atom contributes two electrons. Hence, if the mass fraction of helium is denoted by Y, then the number density of electrons contributed by helium atoms is

$$2\frac{\rho}{4m_{\rm H}}Y = \frac{\rho}{2m_{\rm H}}Y.$$

For each of the heavier elements, i, the number of electrons contributed per atom is equal to the atomic number, Z_i. The mass of each nucleus is $A_i \times m_{\rm H}$. If the mass fraction of an arbitrary heavy element is denoted by Z, then the number density of electrons contributed by this heavy element is

$$Z_i\frac{\rho}{A_i m_{\rm H}}Z \approx \frac{\rho}{2m_{\rm H}},$$

since $Z_i/A_i \approx 1/2$. Hence, if we now denote the total mass fraction of all heavy elements by Z, and note that $X + Y + Z = 1$, then we can write

$$\begin{aligned} n_e &\approx \frac{\rho}{m_{\rm H}}X + \frac{\rho}{2m_{\rm H}}Y + \frac{\rho}{2m_{\rm H}}Z \\ &\approx \frac{\rho}{2m_{\rm H}}\left[2X + 1 - X\right] = \frac{\rho(1+X)}{2m_{\rm H}}. \end{aligned} \tag{12}$$

We can now write

$$\begin{aligned} P &= K_1\rho^{5/3} && \text{Eqn. of state for a nonrelativistic degenerate gas} \\ P &= K_2\rho^{4/3} && \text{Eqn. of state for a relativistic degenerate gas,} \end{aligned} \tag{13}$$

where

$$K_1 = \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{2/3}\left(\frac{1+X}{2m_{\rm H}}\right)^{5/3} \tag{14}$$

and

$$K_2 = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{1+X}{2m_{\rm H}}\right)^{4/3}. \tag{15}$$

Hence, in a fully degenerate gas, the pressure depends only on the density and the chemical composition and is independent of temperature (because we applied the low temperature limit).

Of course, there is not a sharp transition between relativistically degenerate and non-relativistically degenerate gases. Similarly, there is not a sharp transition between an ideal gas and a completely degenerate gas. There is a region of temperature and density in which some intermediate and much more complicated equation of state must be used, a situation known as *partial degeneracy.*

One may wonder why the low-temperature limit is relevant for describing stellar interiors, which are very hot according to our every day standards. The answer to this is that the low temperature limit corresponds to the high density limit. The low-temperature limit is applicable when the Fermi momentum, p_0, is much larger than the classical momentum of an electron provided by its thermal motion, which is

$$m_e v = \sqrt{2m_e E} = \sqrt{3m_e k_{\rm B}T}, \tag{16}$$

i.e. when

$$(3m_e k_{\rm B}T)^{1/2} \ll \frac{h}{2}\left(\frac{3n_e}{\pi}\right)^{1/3} \tag{17}$$

which leads to

$$k_{\rm B}T \ll \frac{h^2}{12m_e}\left(\frac{3n_e}{\pi}\right)^{2/3}. \tag{18}$$

Thus, a quantum gas is a cold gas, but the standard of 'coldness' is set by the density of the gas; a temperature of a billion degrees can be be cold in a very dense gas. Figure 2 illustrates when different pressures matter. In general we have

- Solar-type stars – ideal throughout
- Massive stars – significant contribution of radiation pressure (see below)
- White dwarfs – nonrelativistic radiation pressure

We now consider the stability of the equation of state for degenerate electrons. For an ideal gas of classical (i.e. non-relativistic) particles, the energy of a single particle is $E = mv^2/2 = vp/2$, and hence the internal energy density of the gas is

$$u = \frac{1}{2}\int_0^\infty v\,p\,n(p)\,dp = \frac{3}{2}P \tag{19}$$

where the last equality arises from eqn. (5). In the limiting case of ultra-relativistic particles, we have $E = cp$, where c is the speed of light and hence

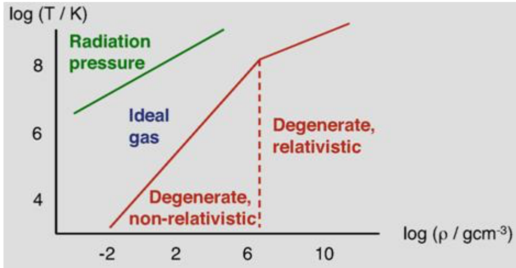

Figure 2:

$$u = \int_0^\infty c\,p\,n(p)\,dp = 3P. \tag{20}$$

From week 4, when we discussed hydrostatic equilibrium and the virial theorem, we showed that the gravitational energy

$$\Omega = -3\int_V P dV, \tag{21}$$

and hence for an ultra-relativistic gas with internal energy density $u = 3P$ we obtain

$$\Omega = -\int_V u dV = -U \tag{22}$$

where U is the total thermal energy, and hence the total energy E is

$$E = \Omega + U = 0. \tag{23}$$

This result tells us that an ultra-relativistic self-gravitating system sits at the point of margin stability: it may either expand or contract indefinitely without any change in the total energy. Hence, a small change to the system may be just enough to push it into instability. Hence, we conclude that systems that evolve towards a state of being supported by ultra-relativistic degeneracy pressure also evolve towards a state of becoming unstable.

1.2 Red giants

When hydrogen becomes exhausted near the centre, a star is left with a core consisting of helium and a small amount of heavy elements. Initially, the temperature in the core is far below the 10^8 K required for helium ignition, and hence there is no nuclear energy generation in the core. Although there may be some release of energy due to gravitational contraction, the luminosity in the core becomes low, and the core becomes almost isothermal.

Surrounding the core is a region containing hydrogen where the temperature is still high enough for hydrogen burning to proceed. This region, which is known as a *hydrogen burning shell source*, provides the energy that powers the luminosity of the star. As the hydrogen is converted into helium in the shell, the mass of the inert helium core increases. This leads to contraction and the release of gravitational energy, half of which goes into the thermal energy of the core. As long as the core is not degenerate, the increase in thermal energy leads to an increase in temperature, up to the point where the temperature in the core is high enough for helium burning to begin (while maintaining the hydrogen shell burning). Once the helium is exhausted, the star has a contracting core consisting of ^{12}C and ^{16}O, which is surrounded by a helium shell source and a hydrogen shell source. If the star is massive enough, contraction of the core may proceed up to the point where the temperature is high enough for carbon ignition, and so on.

This rough description of the evolution deep inside the star ignores a large amount of fascinating detail. Particularly important is the response of the observable properties of the star to the changes in the core: when the region inside the hydrogen burning shell contracts, the region outside the shell expands. This response is behind the dominant observational signature of the post-main sequence evolution, which is rapid expansion of the envelope to form a *red giant* star.

This phenomenon is confirmed by computations of stellar evolution, but so far there is no simple and fully accepted explanation for this phenomenon. While it should be noted that the star increases in it luminosity during this phase, and hence there is an additional amount of thermal energy entering the stellar envelope, there is also another plausible argument. Suppose the core contraction at the end of the hydrogen burning occurs on a time scale shorter than the Kelvin-Helmholtz time scale (this is the time scale during which the star can adjust thermally in a quasi-static fashion to changes in its internal luminosity etc). From energy conservation, we have the sum of the gravitational and thermal energies $E = \Omega + U =$ constant during the core contraction phase (with only a very small amount of energy loss through the stellar surface during this time). From the virial theorem, we have $\Omega + 2U =$ constant. But this is only possible if both Ω and U are conserved separately. The contraction of the core makes the binding energy Ω more negative; this change has to be compensated by the expansion of the envelope. The subsequent evolution of the star depends crucially on its mass, as illustrated by figure 3.

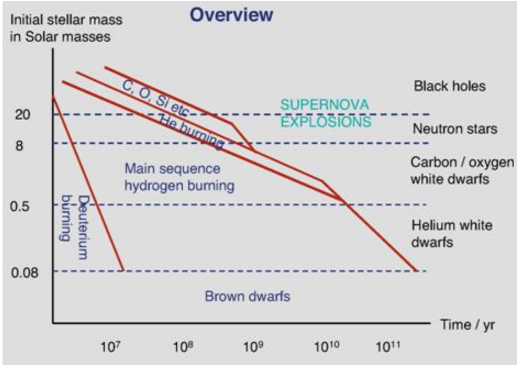

Figure 3:

1.3 White dwarfs

For a star with a contracting core consisting of ^{12}C and ^{16}O, surrounded by a helium and hydrogen shell source, one may expect a repetition of the core contraction leading to sufficiently high temperatures for carbon burning to initiate. A star, however, must contain a mass of more than 8 $M_\odot$ for this to occur. The carbon-oxygen core becomes degenerate, and the electron degeneracy pressure prevents the core from contracting further before the temperature reaches the temperature required for carbon burning.

The subsequent evolution is complex. Numerical computations indicate that a thermal instability develops in the helium burning shell, causing thermal pulses where the star alternates between having a hydrogen and a helium burning shell. At the same time, the luminosity of the star increases dramatically, as does its radius. As a result of the large increase in radius and luminosity, and because of the thermal pulses, the star begins to loose mass at a fairly rapid rate. This process has been called a *superwind*. The result is that the star looses essentially all the material outside the degenerate carbon-oxygen core. The core is initially extremely hot and quite luminous, despite its small size. It illuminates the material that has been lost, which for a few thousand years forms a well-defined shell around the star that shines as a *planetary nebula*. Subsequently, the nebula is dispersed into the interstellar medium, and the degenerate core continues to shine through loss of its thermal energy. It cools gradually, reaching an effective temperature of about 4000 K in 10^{10} years. These objects are called *white dwarfs*. Their masses are typically between 0.5 $M_\odot$ and 1.4 $M_\odot$.

As a white dwarf cools, the pressure generated by the thermal motion of the ions will become less important, and eventually the pressure provided by the degenerate electrons will provide the bulk of the pressure needed to support the star.

We now assume for a while that the star is supported by the pressure of a gas of non-relativistic degenerate electrons (upper expression in eqn. 13). Clearly the density and pressure profiles in such a star are described by a polytropic model ($P = K\rho^\gamma$) with $\gamma = 5/3$ and polytropic index $n = 3/2$ (since $\gamma = 1 + 1/n$). Using eqn. (20) of the week 4 lecture on polytropes

$$K = (4\pi)^{\frac{1}{n}} \frac{G}{n+1} \xi_1^{-\frac{n+1}{n}} \left(-\frac{d\theta}{d\xi} \right)_{\xi=\xi_1}^{\frac{1-n}{n}} M^{\frac{n-1}{n}} R^{\frac{3-n}{n}} . \tag{24}$$

gives the mass-radius relation for an $n = 3/2$ polytrope

$$R \propto M^{-1/3}, \tag{25}$$

demonstrating that more massive white dwarfs have smaller radii. The value of K is given by eqn (14) for a cold, fully degenerate non-relativistic electron gas, and assuming $X = 0$ for simplicity we obtain

$$R = \frac{R_\odot}{74}\left(\frac{M_\odot}{M}\right)^{1/3}. \tag{26}$$

Hence, a solar mass white dwarf has a radius $< 10,000$ km. i.e A bit larger than the Earth!

In many white dwarfs, the electron gas is relativistic in the central part of the star, while it is non-relativistic further out. Indeed, the degenerate electrons become relativistic when the Fermi momentum becomes large compared to $m_e c$, i.e. when the number density of electrons becomes large compared with $(m_e c/h)^3$ (see eqn 8).

If we now consider the limit when the entire white dwarf is filled with ultra-relativistic electrons, the equation of state $P = K_1 \rho^{5/3}$ (upper expression in eqn. 13) should be replaced by $P = K_2 \rho^{4/3}$, with $\gamma = 4/3$ and $n = 3$. For such a polytrope, eqn. (24) predicts a unique mass that has no radius dependence and only depends on K, the polytropic constant. This mass is known as the Chandrasekhar mass, M_{Ch}. We can determine the mass from the equations (15) and (24), and for $X = 0$ we obtain

$$M_{\mathrm{Ch}} = 1.44\,\mathrm{M}_\odot.$$

According to our discussion earlier about the (in)stability of self-gravitating systems with ultra-relativistic degenerate equations of state, we interpret M_{Ch} as being the maximum mass that a white dwarf can have before degeneracy pressure can no longer support the object against gravitational contraction. A white dwarf with mass larger than M_{Ch} will undergo gravitational collapse, and we need additional physics to understand what the fate of such an object might be.

In summary, for lower mass white dwarfs, we expect their electrons to be non-relativistic and for them to have structures that are equivalent to $n = 3/2$ polytropes. As the mass of such an object increases, its density increases and the number of density of the electrons also increases, such that the Fermi momentum increases and eventually the electrons become relativistic. Once the mass exceeds $M_{\mathrm{Ch}} = 1.44\,\mathrm{M}_\odot$, the electrons become ultra-relativistic and such a white dwarf cannot be supported any more by degeneracy pressure and it collapses.