Week 4

1 Polytropic models

One of the primary aims of this module is to derive, from first principles, the full set of equations that describe stellar structure. This will result in a set of differential equations and an equation of state that describe how $\rho(r)$, $P(r)$, $T(r)$, $L(r)$, and $m(r)$ depend on the independent variable r. Hence we will be able, in principle at least, to compute the internal structure of stellar models. So far, we have obtained an equation that describes how $\rho(r)$ and $m(r)$ are related

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r), \tag{1}$$

the hydrostatic equation

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{Gm(r)}{r^2} \tag{2}$$

and the ideal gas equation of state

$$P = \frac{\mathcal{R}}{\mu}\rho T = \frac{k_{\rm B}}{\mu m_{\rm H}}\rho T. \tag{3}$$

It should be clear that without an equation that describes how $T(r)$ varies with r, we currently have 3 equations and 4 unknowns ($P(r)$, $\rho(r)$, $m(r)$ and $T(r)$), and hence our equation set is not complete and cannot be solved. In the lecture of week 2, however, we derived an equation of state that applies to adiabatic changes

$$P = K\rho^{\gamma}, \tag{4}$$

and when K is a constant this results in the pressure, P, being a function of ρ only. Hence, we are able to solve the equations of stellar structure that we have derived so far for such an equation of state.

The equation of state in eqn. (4) with K being a constant is more generally referred to as a *polytropic* relation, and the resulting stellar models are known as *polytropic models* or *polytropes*. These simpler models have played an important role in the development of the subject, as they can solved relatively easily (sometimes analytically as we will see below), and nonetheless are not too dissimilar from realistic models. In particular, where there is a near-adiabatic relation between density and pressure, as expressed in eqn. (4), these models can in fact be rather accurate.

To obtain the equation satisfied by polytropic models, we note that eqn. (2) can be manipulated and differentiated to give

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr}. \tag{5}$$

Hence, using eqns. (1) and (4) we eliminate $m(r)$ and obtain

$$\frac{d}{dr}\left(r^2 K\gamma\rho^{\gamma-2}\frac{d\rho}{dr}\right) = -4\pi G\rho r^2. \tag{6}$$

It is convenient to replace γ by the *polytropic index*, n, defined by

$$\gamma = 1 + \frac{1}{n}, \;\; n = \frac{1}{\gamma - 1}. \tag{7}$$

We also introduce a dimensionless measure of density, θ, through

$$\rho = \rho_{\rm c}\theta^n, \tag{8}$$

where $\rho_{\rm c}$ is the central density. Then eqn. (6) becomes

$$\frac{d}{dr}\left(r^2 K\left[\frac{n+1}{n}\right]\rho_{\rm c}^{(1/n-1)}\theta^{1-n}\rho_{\rm c} n\theta^{n-1}\frac{d\theta}{dr}\right) = -4\pi G\rho_{\rm c}\theta^n r^2 \tag{9}$$

which simplifies to

$$\frac{(n+1)K\rho_c^{(1/n-1)}}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n. \tag{10}$$

To simplify the equation further, we introduce a new measure for the distance to the centre, ξ, defined by

$$r = \alpha\xi, \text{ where } \alpha^2 = \frac{(n+1)K\rho_c^{(1/n-1)}}{4\pi G}. \tag{11}$$

Then the equation finally becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n. \tag{12}$$

This equation is called the *Lane-Emden equation*, and the solution $\theta(\xi)$ is called the *Lane-Emden function.* From eqn. (8) it follows that θ must satisfy the boundary conditions

$$\begin{aligned} \theta(\xi) &= 1 \text{ for } \xi = 0 \\ \frac{d\theta}{d\xi} &= 0 \text{ for } \xi = 0. \end{aligned} \tag{13}$$

The second boundary condition comes from consideration of the hydrostatic equation (2) near the centre of the star

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2} \approx -\frac{G4\pi r^3\rho_c^2}{3r^2} = -\frac{4\pi G\rho_c^2 r}{3}. \tag{14}$$

It is clear that

$$\lim_{r\to 0}\frac{4\pi G\rho_c^2 r}{3} = 0 \quad \text{hence} \quad \left.\frac{dP}{dr}\right|_{r=0} = 0. \tag{15}$$

Given the relations $P = K\rho^{1+1/n}$ and $\rho = \rho_c\theta^n$, it is easy to show that the second boundary condition applies at the centre of the star. We also note that the surface of the model is defined by the point $\xi = \xi_1$ where $\theta(\xi_1) = 0$. i.e. where the density goes to zero. Hence we see that θ takes values $0 \le \theta \le 1$, since the density is expected to be a monotonically decreasing function of radius in a star.

Given the solution $\theta(\xi)$, we can obtain relations between the various quantities characterising the model. It follows immediately from eqn. (11) that the surface radius of the model is

$$R = \alpha\xi_1 = \left[\frac{(n+1)K\rho_c^{(1/n-1)}}{4\pi G}\right]^{1/2}\xi_1. \tag{16}$$

The mass, $m(\xi)$, interior to ξ may be obtained by integrating eqn. (1), using eqns. (8), (11) and (12), giving

$$\begin{aligned} m(\xi) &= \int_0^{\alpha\xi} 4\pi r^2\rho dr = 4\pi\alpha^3\rho_c\int_0^{\xi}\xi^2\theta^n d\xi \\ &= -4\pi\alpha^3\rho_c\int_0^{\xi}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)d\xi \\ &= -4\pi\alpha^3\rho_c\xi^2\frac{d\theta}{d\xi}. \end{aligned} \tag{17}$$

Using eqn. (11) for α, we finally obtain

$$m(\xi) = -4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2}\rho_c^{\frac{3-n}{2n}}\xi^2\frac{d\theta}{d\xi}. \tag{18}$$

In particular, the total mass is given by

$$M = -4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2}\rho_c^{\frac{3-n}{2n}}\left(\xi^2\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}. \tag{19}$$

From eqns. (16) and (19), by eliminating ρ_c, we may find a relation between M, R and K. The result is

$$K = (4\pi)^{\frac{1}{n}} \frac{G}{n+1} \xi_1^{-\frac{n+1}{n}} \left(-\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}^{\frac{1-n}{n}} M^{\frac{n-1}{n}} R^{\frac{3-n}{n}}. \tag{20}$$

There are two different interpretations of this relation. If the constant K in eqn. (3) is given in terms of basic physical constants, and hence is known, eqn. (20) defines a relation between the mass and radius of the star. If, on the other hand, eqn. (3) just expressed proportionality, with the constant K being essentially arbitrary, then eqn (20) may be used to determine K for a star with a given mass and radius. As shown below, one may then determine other quantities for the star. In the former case, therefore, there is a unique polytropic model for a given mass, whereas in the latter case a model can be constructed for any value of M and R.

From the last of eqns. (17) we find that the mean density of the star is

$$\bar{\rho} = \frac{3M}{4\pi R^3} = -\frac{3}{\xi} \left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \rho_c, \tag{21}$$

and hence the central density is determined by the mass and radius as

$$\rho_c = -\frac{\xi_1}{3} \left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}^{-1} \frac{3M}{4\pi R^3} \equiv a_n \frac{3M}{4\pi R^3}, \tag{22}$$

where the last equation defines the constant a_n, which depends on the polytropic index, n, only. Finally, using the equation of state from eqn. (4), with $\gamma = 1 + 1/n$,

$$P_c = K \rho_c^{1+1/n}, \tag{23}$$

and using eqns. (20) and (22), we find that

$$P_c = \frac{1}{4\pi(n+1)} \left(-\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}^{-2} \frac{GM^2}{R^4} \equiv c_n \frac{GM^2}{R^4}, \tag{24}$$

where c_n depends on the polytropic index, n, only. The pressure throughout the model is then determined by

$$P = P_c \theta^{n+1}. \tag{25}$$

When the temperature is related to pressure and density through the ideal gas law, $P = k_B \rho T/(\mu m_H)$, it may be determined from eqns. (8) and (25) as

$$T = T_c \theta, \tag{26}$$

where

$$T_c = \left[(n+1)\xi_1 \left(-\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}\right]^{-1} \frac{GM\mu m_H}{k_B R} \equiv b_n \frac{GM\mu m_H}{k_B R}, \tag{27}$$

where b_n depends on the polytropic index only. In the case when the star is composed of an ideal gas, therefore, θ is a measure of the temperature.

To determine the structure of a polytropic star completely, we only need to find the solution to the Lane-Emden equation (12) with the boundary conditions eqn. (13). Unfortunately, in general no analytical solution is possible. The only exceptions are

$$n = 0: \quad \theta = 1 - \xi^2/6 \qquad \xi_1 = \sqrt{6}$$

$$n = 1: \quad \theta = \frac{\sin\xi}{\xi} \qquad \xi_1 = \pi$$

$$n = 5: \quad \theta = \left(1 + \frac{\xi^2}{3}\right)^{-1/2} \qquad \xi_1 = \infty$$

n	ξ_1	a_n	b_n	c_n
0	2.449	1.00	0.5	0.12
1	3.142	3.29	0.5	0.39
1.5	3.654	5.99	0.54	0.77
2	4.353	11.40	0.60	1.64
3	6.897	54.18	0.85	11.05
4	14.97	662.4	1.67	247.6

Table 1: Properties of polytropic models. Constants a_n, b_n and c_n specify the central density, temperature and pressure as given by eqns. (8), (27) and (24).

ξ	θ	$d\theta/d\xi$		ξ	θ	$d\theta/d\xi$
0	1	0		0	1	0
0.5	0.96	-0.16		0.5	0.96	-0.16
1.0	0.85	-0.29		1.0	0.86	-0.25
1.5	0.68	-0.36		1.5	0.72	-0.28
2.0	0.50	-0.37		2.0	0.58	-0.26
2.5	0.32	-0.34		3.0	0.36	-0.18
3.0	0.16	-0.28		4.0	0.21	-0.12
3.5	0.03	-0.22		6.0	0.04	-0.06
3.654	0	-0.20		6.897	0	-0.04

Table 2: Properties of polytropes of indices $n = 1.5$ (left three columns) and $n = 3$ (right three columns).

The solution for $n = 5$ is evidently peculiar, in that it has infinite radius. However, since

$$\lim_{\xi\to\infty}\left(-\xi^2\frac{d\theta}{d\xi}\right) = \sqrt{27}/3, \quad n = 5 \tag{28}$$

is finite, so is the mass of the model. It may be shown that only for $n > 5$ does the Lane-Emden equation have solutions corresponding to finite radius.

For values of n other than 0, 1 and 5, the Lane-Emden equation must be solved numerically. Extensive tables of the solutions exist, and obtaining numerical solutions is a relatively simple numerical problem. Table 1 lists a number of useful quantities, which enter into the expressions given above, for a selection of polytropic models.

Table 2 presents the solution for two particular cases, $n = 1.5$ and $n = 3$, at selected values of ξ. From table 1 it follows that the properties of polytropic models vary widely with n. This is true in particular of the degree of central condensation, as measured by a_n, the ratio of the central to mean density. For $n = 0$ it is obvious from eqn. (8) that density is constant, and hence $a_1 = 1$, whereas the value of a_n tends to infinity as $n \to 5$. For stars on the main sequence, the central condensation is typically around 100, corresponding to a polytrope of index ≈ 3.3.

It should be noticed also that eqn (24) for the central pressure, and in the ideal gas case eqn. (27) for the central temperature, confirm the simple scaling derived in the previous lecture. Now, however, the polytropic relations contain the additional numerical constants b_n and c_n. It is obvious from table 1 that c_n varies strongly with n; hence our estimate of the central pressure in the lecture of week 3 is at best a rough estimate. On the other hand, the range of variation of b_n is much more modest, except when n is close to the critical case $n = 5$. Thus, the estimate of the central temperature obtained in the lecture of week 3 is reasonable for a broad range of models.

1.1 Gravitational potential energy of a polytropic star

$$\Omega = -\int_0^M \frac{Gm(r)}{r} dm(r). \tag{29}$$

From the polytropic equation of state $P = K\rho^{1+\frac{1}{n}}$ we obtain

$$dP = \left(1 + \frac{1}{n}\right) K\rho^{\frac{1}{n}} d\rho \tag{30}$$

and hence

$$\frac{dP}{\rho} = \left(\frac{n+1}{n}\right) K\rho^{\frac{1}{n}-1} d\rho. \tag{31}$$

From the equation of state we have

$$\frac{P}{\rho} = K\rho^{\frac{1}{n}} \tag{32}$$

and differentiating this expression gives

$$d\left(\frac{P}{\rho}\right) = \frac{1}{n} K\rho^{\frac{1}{n}-1} d\rho. \tag{33}$$

Hence we obtain

$$\frac{dP}{\rho} = (n+1) d\left(\frac{P}{\rho}\right). \tag{34}$$

We will also use the equations that govern the mass distribution and hydrostatic equilibrium:

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho(r) \tag{35}$$

and

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2} \rho(r). \tag{36}$$

We start by writing the gravitational potential energy as follows, and then proceed by integrating by parts a number of times (note that the limits of integration c and s denote the centre and surface, respectively):

$$\Omega = -\int_0^M \frac{Gm(r)}{r} dm(r) = -\frac{1}{2}\int_c^s \frac{Gd(m(r)^2)}{r}. \tag{37}$$

Integrating by parts (noting that the total mass is denoted by M and the radius of the star is given by R) gives

$$\begin{aligned} \Omega &= -\frac{1}{2}\int_c^s \frac{Gd(m(r)^2)}{r} = \left[\frac{Gm(r)^2}{2r}\right]_c^s - \frac{1}{2}\int_c^s \frac{Gm(r)^2}{r^2} dr \\ &= -\frac{GM^2}{2R} + \frac{1}{2}\int_c^s m(r)\frac{dP}{dr}\frac{1}{\rho} dr \\ &= -\frac{GM^2}{2R} + \frac{1}{2}\int_c^s m(r)\frac{dP}{\rho} \\ &= -\frac{GM^2}{2R} + \frac{(n+1)}{2}\int_c^s m(r) d\left(\frac{P}{\rho}\right). \end{aligned} \tag{38}$$

Now we integrate the 2nd term above by parts:

$$\begin{aligned} \Omega &= -\frac{GM^2}{2R} + \left[\frac{(n+1)}{2} m(r)\frac{P}{\rho}\right]_c^s - \frac{(n+1)}{2}\int_c^s \frac{P}{\rho} dm(r) \\ &= -\frac{GM^2}{2R} - \frac{(n+1)}{2}\int_c^s \frac{P}{\rho} 4\pi r^2 \rho(r) dr \\ &= -\frac{GM^2}{2R} - \frac{(n+1)}{2}\int_c^s P\frac{4\pi}{3} d(r^3). \end{aligned} \tag{39}$$

Now we integrate the 2nd term above by parts:

$$\begin{aligned}\Omega &= -\frac{GM^2}{2R} - \left[\frac{(n+1)}{2}\frac{4\pi}{3}Pr^3\right]_c^s + \frac{(n+1)}{6}\int_c^s 4\pi r^3 \frac{dP}{dr}dr \\ &= -\frac{GM^2}{2R} - \frac{(n+1)}{6}\int_c^s 4\pi r^3 \frac{Gm(r)}{r^2}\rho(r)dr \\ &= -\frac{GM^2}{2R} - \frac{(n+1)}{6}\int_c^s \frac{Gm(r)}{r}dm(r). \end{aligned} \tag{40}$$

The above expression can be written as

$$\Omega = -\frac{GM^2}{2R} + \frac{(n+1)}{6}\Omega. \tag{41}$$

Solving for Ω in equation (41) finally gives the expression for the gravitational potential energy

$$\Omega = -\frac{3}{5-n}\frac{GM^2}{R}. \tag{42}$$

Hence, we see that for an $n = 0$ polytrope, which is of constant density, the value for the gravitational potential energy agrees with that obtained in the lecture notes of week 3. As the value of n increases towards $n = 5$, and the degree of central concentration increases, the value of Ω becomes increasingly negative until it becomes undefined for $n = 5$.

1.2 Solving the Lane-Emden equation numerically

We recall that the Lane-Emden equation is

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \tag{43}$$

and the boundary conditions at the centre of the star are

$$\theta(\xi = 0) = 1 \text{ and } \left(\frac{d\theta}{d\xi}\right)_{\xi=0} = 0, \tag{44}$$

where θ^n plays the role of a scaled density and ξ plays the role of the scaled radius when the equation of state is expressed as $P = K\rho^{1+1/n}$.

Note that we have the following relations

$$\begin{aligned}\rho &= \rho_c\theta^n \\ P &= P_c\theta^{n+1} \\ T &= T_c\theta^n. \end{aligned} \tag{45}$$

We also have

$$\begin{aligned}\rho_c &= a_n\frac{3M}{4\pi R^3} \quad \text{where} \quad a_n = -\frac{\xi_1}{3}\left(\frac{d\theta}{d\xi}\right)^{-1}_{\xi=\xi_1} \\ P_c &= c_n\frac{GM^2}{R^4} \quad \text{where} \quad c_n = \frac{1}{4\pi(n+1)}\left(-\frac{d\theta}{d\xi}\right)^{-2}_{\xi=\xi_1} \\ T_c &= b_n\frac{GM}{R}\frac{\mu m_H}{k_B} \quad \text{where} \quad b_n = \left[(n+1)\xi_1\left(\frac{-d\theta}{d\xi}\right)_{\xi=\xi_1}\right]^{-1}. \end{aligned} \tag{46}$$

We note that ξ_1 corresponds to the value of ξ where $\theta(\xi) = 0$. We also have the relation $r = \alpha\xi$ where

$$\alpha = \sqrt{\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}}$$

and

$$K = (4\pi)^{1/n} \frac{G}{(n+1)} \xi_1^{-(n+1)/n} \left(-\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}^{(1-n)/n} M^{(n-1)/n} R^{(3-n)/n}.$$

Hence, if we specify the total stellar mass, M, and the stellar radius, R, and solve for $\theta(\xi)$, $(-d\theta/d\xi)_{\xi=\xi_1}$ and ξ_1 we can determine how ρ, P and T vary with radius.

1.2.1 A simple numerical algorithm

We now present a sequence of steps that allow us to solve the Lane-Emden equation numerically.

(1). Rewrite eqn. (43) in the form

$$\frac{d^2\theta}{d\xi^2} = -\frac{2}{\xi}\frac{d\theta}{d\xi} - \theta^n. \tag{47}$$

(2). Approximate derivatives of the function θ as

$$\frac{d\theta}{d\xi} \approx \frac{\Delta\theta}{\Delta\xi}$$

where $\Delta\theta$ represents a small but finite change in θ, and $\Delta\xi$ is a small but finite change in ξ. Using the same approximation, we can write

$$\frac{d^2\theta}{d\xi^2} \equiv \frac{d}{d\xi}\left(\frac{d\theta}{d\xi}\right) \approx \frac{1}{\Delta\xi} \times \Delta\left(\frac{d\theta}{d\xi}\right)$$

where $\Delta\left(d\theta/d\xi\right)$ represents a small but finite change in $d\theta/d\xi$.

(3). Now divide the radius of the star into discrete points, starting with a central value ξ_0, and with successive points ξ_1, ξ_2, ..., ξ_i, ξ_{i+1}, ... (note that ξ_1 here does not correspond to the surface of the model, as it does in our earlier discussion).

Define $\Delta\xi = \xi_{i+1} - \xi_i$, the distance between any two successive points.

Note that when we solve the Lane-Emden equation we will compute the values of θ and $d\theta/d\xi$ at the discrete points. These will be denoted as θ_0, θ_1, θ_2, ..., θ_i and $(d\theta/d\xi)_0$, $(d\theta/d\xi)_1$, $(d\theta/d\xi)_2$, ..., $(d\theta/d\xi)_i$, ...

(4). Now rewrite the Lane-Emden equation as

$$\frac{1}{\Delta\xi} \times \Delta\left(\frac{d\theta}{d\xi}\right) = -\frac{2}{\xi}\frac{d\theta}{d\xi} - \theta^n \tag{48}$$

which then becomes

$$\frac{1}{\Delta\xi}\left[\left(\frac{d\theta}{d\xi}\right)_{i+1} - \left(\frac{d\theta}{d\xi}\right)_i\right] = -\frac{2}{\xi_i}\left(\frac{d\theta}{d\xi}\right)_i - \theta_i^n \tag{49}$$

with the subscripts denoting the fact that these quantities are defined at discrete points in scaled radius ξ_i. Equation (49) can now be written as

$$\left(\frac{d\theta}{d\xi}\right)_{i+1} = \left(\frac{d\theta}{d\xi}\right)_i - \Delta\xi\left[\frac{2}{\xi_i}\left(\frac{d\theta}{d\xi}\right)_i + \theta_i^n\right] \tag{50}$$

Using the approximation

$$\left(\frac{\Delta\theta}{\Delta\xi}\right)_{i+1} = \frac{\theta_{i+1} - \theta_i}{\Delta\xi} \approx \left(\frac{d\theta}{d\xi}\right)_{i+1} \tag{51}$$

we can now write

$$\theta_{i+1} = \theta_i + \Delta\xi\left(\frac{d\theta}{d\xi}\right)_{i+1}. \tag{52}$$

Equations (51) and (52) form a set of coupled equations that can be used to solve the Lane-Emden equation

$$\left(\frac{d\theta}{d\xi}\right)_{i+1} = \left(\frac{d\theta}{d\xi}\right)_i - \Delta\xi\left[\frac{2}{\xi_i}\left(\frac{d\theta}{d\xi}\right)_i + \theta_i^n\right]$$

$$\theta_{i+1} = \theta_i + \Delta\xi\left(\frac{d\theta}{d\xi}\right)_{i+1}$$

$$\text{with the boundary conditions} \qquad \theta_0 = 1 \text{ and } \left(\frac{d\theta}{d\xi}\right)_0 = 0 \qquad (53)$$

(5). We now solve eqns. (53) using the following steps

i. Define $\xi_0 = 10^{-6}$. We should use $\xi_0 = 0.0$ to define the radius point at the centre of the star, but this would introduce a division by zero in the first of equations (53) so we choose a small number close to zero instead.

ii. Define a value for $\Delta\xi$. We will use $\Delta\xi = 10^{-3}$.

iii. Impose the boundary conditions by defining $\theta_0 = 1.0$ and $(d\theta/d\xi)_0 = 0.0$

iv. Using the equations (53), start stepping out from the centre of the star, updating

$$\left(\frac{d\theta}{d\xi}\right)_{i+1}, \quad \theta_{i+1} \text{ and } \xi_{i+1}$$

at each radius point. The first update will be

$$\left(\frac{d\theta}{d\xi}\right)_1 = \left(\frac{d\theta}{d\xi}\right)_0 - \Delta\xi\left[\frac{2}{\xi_0}\left(\frac{d\theta}{d\xi}\right)_0 + \theta_0^n\right]$$

$$\theta_1 = \theta_0 + \Delta\xi\left(\frac{d\theta}{d\xi}\right)_1$$

$$\xi_1 = \xi_0 + \Delta\xi$$

The second update will be

$$\left(\frac{d\theta}{d\xi}\right)_2 = \left(\frac{d\theta}{d\xi}\right)_1 - \Delta\xi\left[\frac{2}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_1 + \theta_1^n\right]$$

$$\theta_2 = \theta_1 + \Delta\xi\left(\frac{d\theta}{d\xi}\right)_2$$

$$\xi_2 = \xi_1 + \Delta\xi$$

v. Use eqns. (53) to continue stepping out towards the surface of the star. Stop stepping out when the value of $\theta_{i+1} < 0$ as this indicates that we have gone beyond the surface of the star where $\theta = 0$.

vi. Use the final values of ξ_i, ξ_{i+1}, θ_i θ_{i+1}, $(d\theta/d\xi)_i$ and $(d\theta/d\xi)_{i+1})$ to interpolate and find the values of $\xi = \xi_1$ and $(d\theta/d\xi)_{\xi=\xi_1}$ that correspond to the point where $\theta = 0$.

(6). We have now obtained values for ξ_1 and $(d\theta/d\xi)_{\xi=xi_1}$, and combined with specified values for M and R they can be used to define the constants a_n, b_n and c_n, and hence ρ_c, P_c and T_c.

(7). We also have values for θ_i at numerous positions inside the star, and hence we can define the values of ρ, P and T at these positions. By using M and R to calculate K we can define α and therefore the values of r that correspond to the ξ_i values. Hence we now know how ρ, P and T depend on r. An example of a $n = 3.3$ polytrope for a solar mass and radius star is shown in Fig. 1, and compared with the Standard Solar model which can be downloaded from http://www.sns.ias.edu/ jnb/SNdata/Export/BS2005/bs05op.dat.

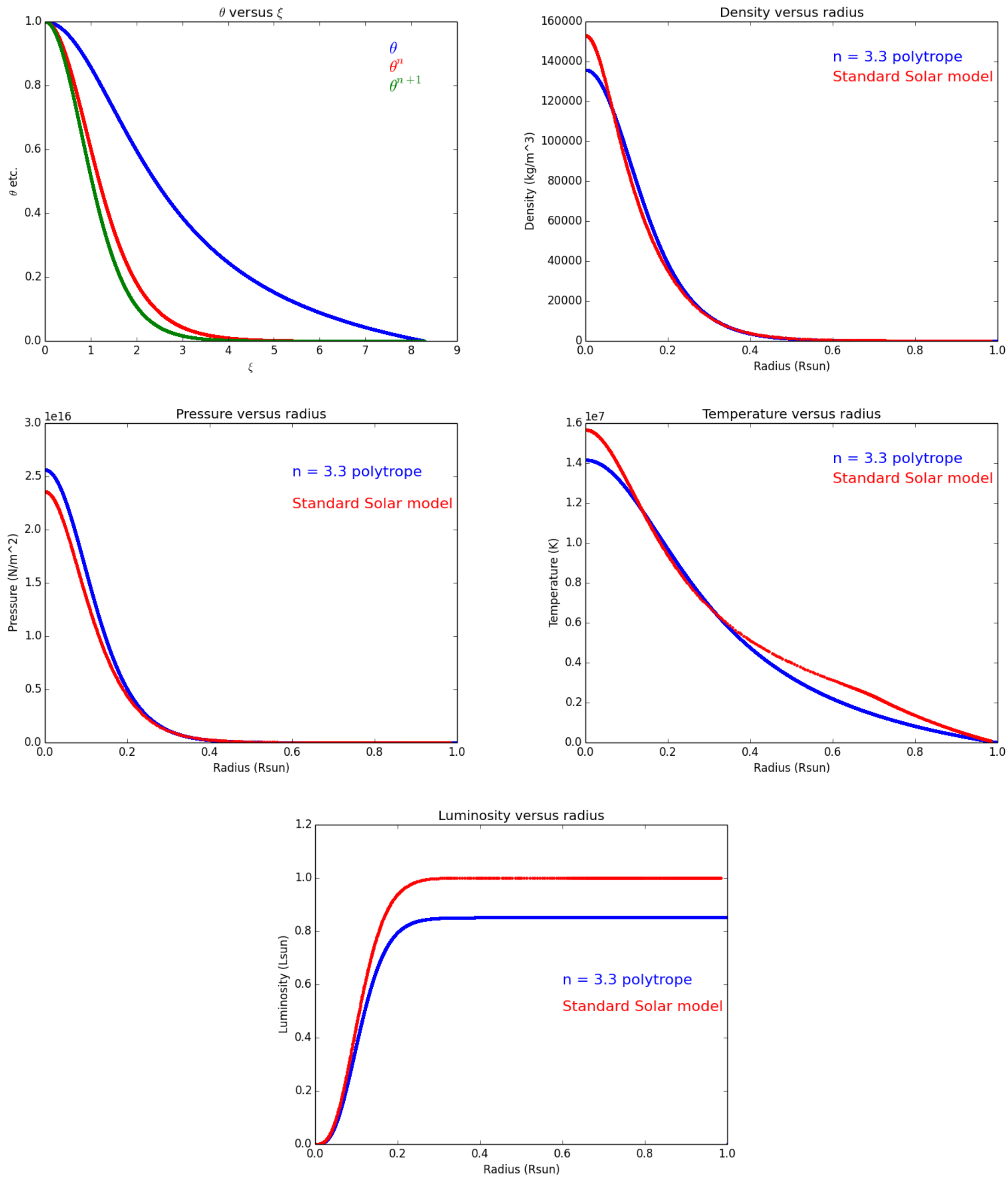

Figure 1: These plots compare the values of ρ, P, T and L obtained for an $n = 3.3$ polytrope with the Standard Solar Moodel (see http://www.sns.ias.edu/ jnb/SNdata/Export/BS2005/bs05op.dat). Note that the luminosity obtained for the polytrope was calculated using the expression derived for $\epsilon_{\rm pp}$ presented in the lecture of week 5: $\epsilon_{\rm pp} = 2.6 \times 10^{-37} X^2 \rho T^4$. Note the modified exponent on the temperature which was required to obtain decent agreement with the Solar model.