1B40 Practical Skills

Combining uncertainties from several quantities – error propagation

We usually encounter situations where the result of an experiment is given in terms of two (or more) quantities. We then need to know what is the error on the final answer, in terms of the uncertainties on the individual quantities. We will discuss some specific cases before dealing with a more general case; a summary appears at the end.

Linear cases

A simple case is where the result, *Z*, is a linear sum of two other quantities *A* and *B*, e.g. $z = a - b.$ If *A* has the value $a \pm \delta a$ and *B* the value $b \pm \delta b$ then *Z* has the value $z \pm \delta z.$ How is δz related to δa and δb? We might expect that $\delta z = \delta a - \delta b.$ However this could lead to the nonsense result that $\delta z \leq 0$! We could decide to consider the maximum possible error and simply add the magnitudes of δa and δb. This would be an overestimation of the error.

A more sensible approach is to consider standard deviations. Thus if we square, $\delta z = \delta a - \delta b$ we get

$$(\delta z)^2 = (\delta a)^2 + (\delta b)^2 - 2\delta a\, \delta b.$$

Over a large number of measurements we would expect that if δa and δb are both equally positive and negative and **not correlated to each other** then the average value of $\delta a \delta b$ is zero. Hence

$$\left\langle (\delta z)^2 \right\rangle = \left\langle (\delta a)^2 \right\rangle + \left\langle (\delta b)^2 \right\rangle,$$

$$\sigma_z^2 = \sigma_a^2 + \sigma_b^2,$$

and so we add the errors on *a* and *b* in quadrature. Note the result would have been the same if $z = a + b.$

A formal derivation of the result is as follows,

$$\sigma_z^2 = \left\langle \left[z - \bar{z} \right]^2 \right\rangle = \left\langle \left[(a - b) - (\bar{a} - \bar{b}) \right]^2 \right\rangle,$$

$$\sigma_z^2 = \left\langle \left[(a - \bar{a}) - (b - \bar{b}) \right]^2 \right\rangle,$$

$$\sigma_z^2 = \left\langle (a - \bar{a})^2 \right\rangle + \left\langle (b - \bar{b})^2 \right\rangle - 2\left\langle (a - \bar{a})(b - \bar{b}) \right\rangle,$$

$$\sigma_z^2 = \sigma_a^2 + \sigma_b^2 - 2\,\mathrm{cov}(a, b).$$

The last term involves the covariance of *a* and *b*. This is a measure of whether their errors are correlated or not. It can be positive or negative or, in the case where they are uncorrelated, zero. Its value is related to the extent that a value of δa affects that of $\delta b.$

Products and quotients

If $z = ab$ then

$$z + \delta z = (a + \delta a)(b + \delta b) = ab + a\delta b + b\delta a + \delta a\delta b,$$

$$\delta z = a\delta b + b\delta a + \delta a\delta b.$$

To first-order in the errors,

$$\delta z = a\delta b + b\delta a.$$

Since we don't know the signs of the uncertainties, we square and average over a large number of measurements,

$$(\delta z)^2 = a^2(\delta b)^2 + b^2(\delta a)^2 + 2ab(\delta a)(\delta b),$$

$$\left\langle(\delta z)^2\right\rangle = a^2\left\langle(\delta b)^2\right\rangle + b^2\left\langle(\delta a)^2\right\rangle + 2ab\left\langle(\delta a)(\delta b)\right\rangle.$$

If the uncertainties in *a* and *b*, δa and δb, are uncorrelated, $\left\langle(\delta a)(\delta b)\right\rangle = 0$ and

$$\left\langle\left(\frac{\delta z}{z}\right)^2\right\rangle = \left\langle\left(\frac{\delta a}{a}\right)^2\right\rangle + \left\langle\left(\frac{\delta b}{b}\right)^2\right\rangle,$$

$$\left(\frac{\sigma_z}{z}\right)^2 = \left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2.$$

Thus in this case we add the fractional errors in quadrature.

If $z = \dfrac{a}{b}$ then

$$z + \delta z = \frac{a + \delta a}{b + \delta b} = \frac{a(1 + \delta a / a)}{b(1 + \delta b / b)},$$

$$z + \delta z \approx \frac{a}{b}\left(1 + \frac{\delta a}{a}\right)\left(1 - \frac{\delta b}{b}\right) = \frac{a}{b}\left(1 + \frac{\delta a}{a} - \frac{\delta b}{b}\right),$$

$$\frac{\delta z}{z} = \frac{\delta a}{a} - \frac{\delta b}{b}.$$

where in the second line we have use the binomial approximation for

$\left(1 + \dfrac{\delta b}{b}\right)^{-1} \approx 1 - \dfrac{\delta b}{b}$ when $\delta b / b \ll 1$.

Just as in the linear case we don't know the sign of the uncertainties, so squaring as before

$$\left(\frac{\delta z}{z}\right)^2 = \left(\frac{\delta a}{a}\right)^2 + \left(\frac{\delta b}{b}\right)^2 - 2\left(\frac{\delta a}{a}\right)\left(\frac{\delta b}{b}\right),$$

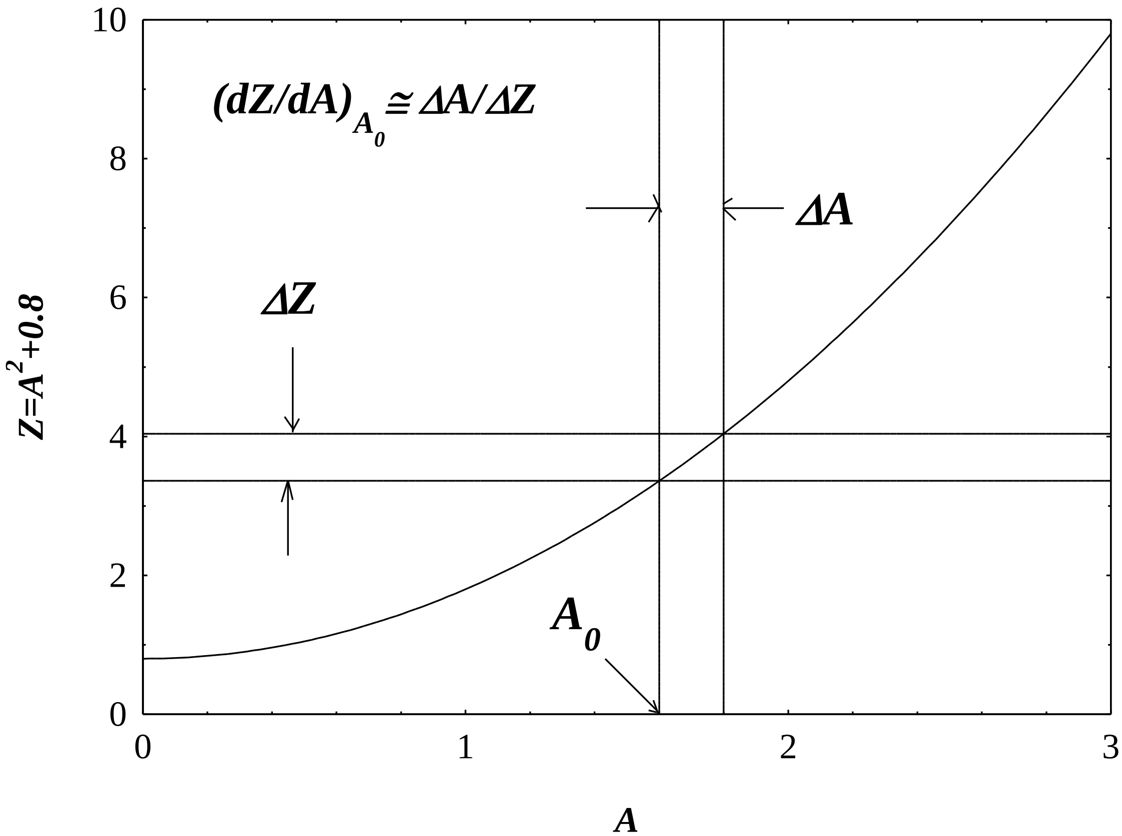
$$\left\langle\left(\frac{\delta z}{z}\right)^2\right\rangle = \left\langle\left(\frac{\delta a}{a}\right)^2\right\rangle + \left\langle\left(\frac{\delta b}{b}\right)^2\right\rangle,$$

$$\left(\frac{\sigma_z}{z}\right)^2 = \left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2.$$

if the errors are uncorrelated. Thus also in this case we add in quadrature the fractional errors.

Functions of a single variable

Suppose we have a measured quantity *A*, and determined its value as $a \pm \delta a$. The reason *A* was measured was to provide an *indirect* measurement of a quantity *Z* which is a known function of *A*, i.e. $z = f(a)$. What is the best estimate for $z \pm \delta z$? The graph illustrates such a case for $z = a^2$.



The uncertainty, δZ, in Z is related to that in *A* by

$$\delta z = \left(\frac{dz}{da} \right)_{a=a_0} \delta a.$$

This is equivalent to assuming that *f(a)* is linear over the small region we are considering and we evaluate the differential at the mean of the measurements. This is a general result. Three examples are

1) $z = a^n \ldots\ldots \frac{dz}{da} = na^{n-1},$ giving $\frac{\delta z}{z} = n\frac{\delta a}{a}.$

2) $z = \ln a \ldots\ldots \frac{dz}{da} = \frac{1}{a},$ giving $\delta z = \frac{\delta a}{a}.$

3) $z = \exp(a) \ldots\ldots \frac{dz}{da} = \exp(a) = z,$ giving $\frac{\delta z}{z} = \delta a.$

Functions of Several Variables

If $z = f(a,b)$ and the uncertainties in A and B are given by δa and δb respectively we make the same approximation as before, i.e. z is a linear function of a and b in the relevant region, so that we can write

$$\delta z = \left(\frac{\partial z}{\partial a}\right)_{A=a_0} \delta a + \left(\frac{\partial z}{\partial b}\right)_{B=b_0} \delta b.$$

The quantity $\frac{\partial z}{\partial a}$ is called a *partial derivative*. You will learn more about these in your mathematics courses (if you haven't already met them). To obtain an expression for $\frac{\partial z}{\partial a}$ we consider z as a function of the variable a, all other quantities are considered as constant, and differentiate the expression for z in the normal way.

As before the signs of the uncertainties are not known, so we square the expression and average over many measurements,

$$(\delta z)^2 = \left(\frac{\partial z}{\partial a}\right)^2 (\delta a)^2 + \left(\frac{\partial z}{\partial b}\right)^2 (\delta b)^2 + 2\left(\frac{\partial z}{\partial a}\right)\left(\frac{\partial z}{\partial b}\right)(\delta a)(\delta b),$$

$$\left\langle(\delta z)^2\right\rangle = \left(\frac{\partial z}{\partial a}\right)^2 \left\langle(\delta a)^2\right\rangle + \left(\frac{\partial z}{\partial b}\right)^2 \left\langle(\delta b)^2\right\rangle + 2\left(\frac{\partial z}{\partial a}\right)\left(\frac{\partial z}{\partial b}\right)\left\langle(\delta a)(\delta b)\right\rangle,$$

$$\left\langle(\delta z)^2\right\rangle = \left(\frac{\partial z}{\partial a}\right)^2 \left\langle(\delta a)^2\right\rangle + \left(\frac{\partial z}{\partial b}\right)^2 \left\langle(\delta b)^2\right\rangle + 2\left(\frac{\partial z}{\partial a}\right)\left(\frac{\partial z}{\partial b}\right)\mathrm{cov}(a,b).$$

The last term is zero if A and B are *independent* variables as there is equal probability of positive and negative combinations of δa and δb whose average is therefore zero. Thus

$$\left\langle(\delta z)^2\right\rangle = \left(\frac{\partial z}{\partial a}\right)^2 \left\langle(\delta a)^2\right\rangle + \left(\frac{\partial z}{\partial b}\right)^2 \left\langle(\delta b)^2\right\rangle,$$

$$\sigma_z^2 = \left(\frac{\partial z}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial z}{\partial b}\right)^2 \sigma_b^2.$$

This can be extended to more variables by adding further similar terms, *viz*

$$\sigma_z^2 = \left(\frac{\partial z}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial z}{\partial b}\right)^2 \sigma_b^2 + \left(\frac{\partial z}{\partial c}\right)^2 \sigma_c^2 + \ldots.$$

All the earlier examples, e.g. $z = a + b$ or $z = a/b$ are just special cases of the general result given above.

Correlated uncertainties

Most of the measurements of quantities you will meet in the laboratory course are uncorrelated, or at least weakly correlated. So the subtle effects of correlations will usually be ignored. However it is worth considering an extreme case to illustrate their effects. Consider the simple case of $z = a + b$. The error on z is given by

$$\sigma_z^2 = \sigma_a^2 + \sigma_b^2.$$

If $b \equiv a$ then this result leads to

$$\sigma_z^2 = 2\sigma_a^2,$$
$$\sigma_z = \sqrt{2}\sigma_a.$$
Yet if we had started from $z = 2a$ we would immediately get $\delta z = 2\delta a$ and hence
$$\sigma_z = 2\sigma_a,$$
in contradiction to the previous result. The **latter is correct**, the former is wrong as it has not allowed for the correlation between a and $b\left(\equiv a\right)$ which is, of course 100%. We have $\left\langle(\delta a)(\delta a)\right\rangle \neq 0$, in fact $\mathrm{cov}(a,a) = 1$.

Example

Consider that a quantity z is dependent on quantities a, b and c in the following way,
$$z = \alpha a^2 + \sqrt{b^2 + \beta \exp\left(c\right)}$$
where α and β are constants. (This is a contrived expression simply for the purposes of this exercise!) We want to find the error on z in terms of those on a, b and c. We can proceed either by applying the formulae for the various special cases considered above, or the general formula.

Case 1 – apply special case formulae.
Let $y = b^2 + \beta \exp\left(c\right)$ and $p = y^{1/2}$, then
$$z = \alpha a^2 + y^{1/2} = \alpha a^2 + p.$$
By differentiation
$$\delta p = \frac{1}{2} y^{-1/2} \delta y,$$
and so
$$\left(\delta z\right)^2 = \left(\alpha 2a\delta a\right)^2 + \left(\delta p\right)^2$$
$$\left(\delta z\right)^2 = 4\alpha^2 a^2 \left(\delta a\right)^2 + \frac{1}{4y}\left(\delta y\right)^2.$$
But
$$\left(\delta y\right)^2 = [\delta\left(b^2\right)] + \beta^2[\delta\left(\exp(c)\right)]^2$$
$$\left(\delta y\right)^2 = 4b^2\left(\delta b\right)^2 + \beta^2 \exp(2c)\left(\delta c\right)^2.$$
Thus combining the results we get
$$\left(\delta z\right)^2 = 4\alpha^2 a^2 \left(\delta a\right)^2 + \frac{1}{4\left(b^2 + \beta \exp(c)\right)}\left[4b^2\left(\delta b\right)^2 + \beta^2 \exp(2c)\left(\delta c\right)^2\right].$$
Case 2 – application of general formula.
By partial differentiation we have

$$\begin{aligned}\frac{\partial z}{\partial a} &= 2\alpha a,\\ \frac{\partial z}{\partial b} &= \frac{1}{2\sqrt{b^2+\exp(c)}}2b,\\ \frac{\partial z}{\partial c} &= \frac{1}{2\sqrt{b^2+\exp(c)}}\beta\exp(c).\end{aligned}$$

The general formula then gives

$$\sigma_z^2 = \left(\frac{\partial z}{\partial a}\right)^2\sigma_a^2 + \left(\frac{\partial z}{\partial b}\right)^2\sigma_b^2 + \left(\frac{\partial z}{\partial c}\right)^2\sigma_c^2,$$

$$\sigma_z^2 = 4\alpha^2 a^2\sigma_a^2 + \frac{1}{4\left(b^2+\beta\exp(c)\right)}\left[4b^2\sigma_b^2 + \beta^2\exp(2c)\sigma_c^2\right].$$

Summary of error propagation formulae

If quantities a and b have errors σ_a and σ_b respectively and are uncorrelated, and k and n are constants, then if

quantity	error on quantity
$z = na$	$\sigma_z = n\sigma_a; \quad \sigma_z = z\left(\frac{\sigma_a}{a}\right)$
$z = a + b$	$\sigma_z^2 = \sigma_a^2 + \sigma_b^2; \qquad \sigma_z = \sqrt{\sigma_a^2 + \sigma_b^2}$
$z = a - b$	$\sigma_z^2 = \sigma_a^2 + \sigma_b^2; \qquad \sigma_z = \sqrt{\sigma_a^2 + \sigma_b^2}$
$z = ab$	$\left(\frac{\sigma_z}{z}\right)^2 = \left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2; \qquad \sigma_z = z\sqrt{\left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2}$
$z = a/b$	$\left(\frac{\sigma_z}{z}\right)^2 = \left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2; \qquad \sigma_z = z\sqrt{\left(\frac{\sigma_a}{a}\right)^2 + \left(\frac{\sigma_b}{b}\right)^2}$
$z = a^n$	$\left(\frac{\sigma_z}{z}\right) = n\left(\frac{\sigma_a}{a}\right); \qquad \sigma_z = zn\left(\frac{\sigma_a}{a}\right) = na^{n-1}\sigma_a$
$z = \ln(a)$	$\sigma_z = \frac{\sigma_a}{a}$
$z = \exp(a)$	$\sigma_z = z\sigma_a = \mathrm{e}^a\sigma_a$
$z = \mathrm{e}^{ka}$	$\sigma_z = zk\sigma_a = \mathrm{e}^{ka}k\sigma_a$
$z = \sin(ka)$	$k\cos(ka)\sigma_a$ Note a and σ_a are in radians!
$z = f(a, b, c, \ldots)$	$\sigma_z^2 = \left(\frac{\partial f}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial f}{\partial b}\right)^2 \sigma_b^2 + \left(\frac{\partial f}{\partial c}\right)^2 \sigma_c^2 + \ldots$

In complex expressions, either apply the general result from the last row, or break the expression down into sums, products and quotients and apply the separate formulae and combine the results.