Cosmology Part II: The Perturbed Universe

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Classes

Thursdays, Spring Term

13/1	Cruciform B.3.01	10.00 – 13.00
20/1	Physics E1	10.00 – 13.00
27/1	Physics E7	10.00 – 13.00
03/2	Physics E1	10.00 – 13.00
10/2	Cruciform B.3.01	09.00 – 12.00 (note different time)
17/2	NO LECTURE	
24/2	Physics E1	10.00 – 13.00
03/3	Physics E1	10.00 – 13.00
10/3	Physics E7	10.00 – 13.00
17/3	Physics E1	10.00 – 13.00
24/3	Physics E1	10.00 – 13.00

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Course Website

`http://zuserver2.star.ucl.ac.uk/~hiranya/PHASM336/`

Introductory Reading:

1. Liddle, A. An Introduction to Modern Cosmology. Wiley (2003)

Complementary Reading

1. Dodelson, S. Modern Cosmology. Academic Press (2003) *
2. Carroll, S.M. Spacetime and Geometry. Addison-Wesley (2004) *
3. Liddle, A.R. and Lyth, D.H. Cosmological Inflation and Large-Scale Structure. Cambridge (2000)
4. Kolb, E.W. and Turner, M.S. The Early Universe. Addison-Wesley (1990)
5. Weinberg, S. Gravitation and Cosmology. Wiley (1972)
6. Peacock, J.A. Cosmological Physics. Cambridge (2000)
7. Mukhanov, V. Physical Foundations of Cosmology. Cambridge (2005)

Books denoted with a * are particularly recommended for this course.

Acknowledgements

These notes borrow with gratitude from excellent notes by (in no particular order) Richard Battye, Anthony Challinor and Wayne Hu. It most closely parallels the treatment found in Dodelson. I am supported by STFC, the European Commission, and the Leverhulme Trust.

Errata

Any errata contained in the following notes are solely my own. Reports of any typos or unclear explanations in the notes will be gratefully received at the email address below. The notes are evolving, and the most up-to-date version at any given time will be found on the website above.

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Contents

I. LARGE SCALE STRUCTURE FORMATION

The real universe is far from homogeneous and isotropic except on the largest scales. Figure 1 shows slices through the 3D distribution of galaxy positions from the 2dF galaxy redshift survey out to a comoving distance of 600 Mpc. The distribution of galaxies is clearly not random; instead they are arranged into a delicate *cosmic web* with galaxies strung out along dense filaments and clustering at their intersections leaving huge empty voids. However, if we smooth the picture on large scales ($\sim 100\,\mathrm{Mpc}$) it starts to look much more homogeneous. Furthermore, we know from the CMB that the universe was smooth to around 1 part in 10^5 at the time of recombination; see Fig. 2. The aim of this part of the course is to study the growth of large-scale structure in an expanding universe through *gravitational instability* acting on small initial perturbations. We shall then learn how these initial perturbations were likely produced by quantum effects during cosmological inflation.

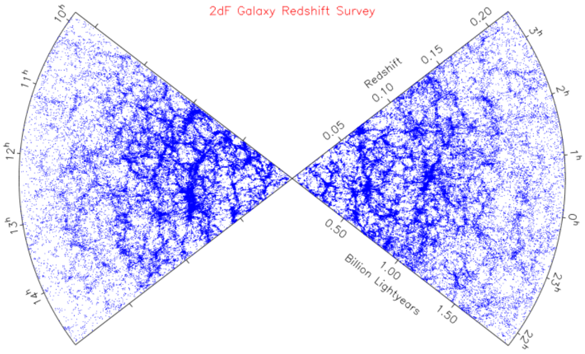

FIG. 1 Slices through the 3D map of galaxy positions from the 2dF galaxy redshift survey. Note that redshift 0.15 is at a comoving distance of 600 Mpc. Figure credit: 2dF.

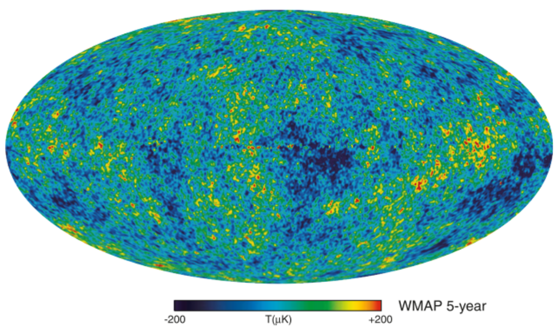

FIG. 2 Fluctuations in the CMB temperature, as determined from five years of WMAP data, about the average temperature of 2.725 K. The fluctuations are at the level of only a few parts in 10^5. Credit: WMAP science team.

A. Overview of structure formation

We will compute $P_i(k)$, the initial power spectrum of density fluctuations, e.g. from inflation. The aim of this section is to understand how this initial spectrum is processed by the evolution of the universe, using linear perturbation theory.

This processing is often quantified in terms of the *transfer function*:

$$\delta_{\mathbf{k}}(t_0) = T(k)\delta_{\mathbf{k}}(t_i) \Rightarrow P(k) = T^2(k)P_i(k)\,. \tag{2.1.1}$$

II. STATISTICS OF RANDOM FIELDS

`READING:` This section, which will not be covered during class, gives the precise mathematical definition of some key concepts: the *power spectrum*, *correlation function* and *angular power spectrum*, that you will need later in the course. These derivations are **NON-EXAMINABLE**, although of course the use of the physical concepts specified above is examinable. You are invited to read the following material to deepen your understanding of the subsequent material. When we come to the relevant concepts later in the course, you will be expected to have a grasp of where they came from and what they mean.

Theory (e.g. quantum mechanics during inflation) only allows us to predict the statistical properties of cosmological fields (such as the matter overdensity $\delta\rho$). Here, we explore the basic statistical properties enforced on such fields by assuming the physics that generates the initial fluctuations, and subsequently processes them, respects the symmetries of the background cosmology, i.e. isotropy and homogeneity.

Throughout, we denote expectation values with angle brackets, e.g. $\langle\delta\rho\rangle$; you should think of this as a quantum expectation value or an average over a classical ensemble[1]. To keep the Fourier analysis simple, we shall only consider flat ($K = 0$) background models and we denote comoving spatial positions by $\mathbf{x}$.

A. Random fields in 3D Euclidean space

Consider a random field $f(\mathbf{x})$ – i.e. at each point $f(\mathbf{x})$ is some random number – with zero mean, $\langle f(\mathbf{x})\rangle = 0$. The probability of realising some field configuration is a *functional* $\Pr[f(\mathbf{x})]$. *Correlators* of fields are expectation values of products of fields at different spatial points (and, generally, times). The two point correlator is

$$\xi(\mathbf{x},\mathbf{y}) \equiv \langle f(\mathbf{x})f(\mathbf{y})\rangle = \int \mathcal{D}f \,\Pr[f]f(\mathbf{x})f(\mathbf{y})\,, \tag{2.2.1}$$

where the integral is a *functional integral* (or path integral) over field configurations.

Statistical homogeneity means that the statistical properties of the translated field,

$$\hat{T}_{\mathbf{a}}f(\mathbf{x}) \equiv f(\mathbf{x}-\mathbf{a})\,, \tag{2.2.2}$$

are the same as the original field, i.e. $\Pr[f(\mathbf{x})] = \Pr[\hat{T}_{\mathbf{a}}f(\mathbf{x})]$. For the two-point correlation, this means that

$$\begin{aligned} &\xi(\mathbf{x},\mathbf{y}) = \xi(\mathbf{x}-\mathbf{a},\mathbf{y}-\mathbf{a}) \quad \forall\mathbf{a} \\ \Rightarrow\quad &\xi(\mathbf{x},\mathbf{y}) = \xi(\mathbf{x}-\mathbf{y})\,, \end{aligned} \tag{2.2.3}$$

so the two-point correlator only depends on the separation of the two points.

Statistical isotropy mean that the statistical properties of the rotated field,

$$\hat{R}f(\mathbf{x}) \equiv f(\mathsf{R}^{-1}\mathbf{x})\,, \tag{2.2.4}$$

where R is a rotation matrix, are the same as the original field, i.e. $\Pr[f(\mathbf{x})] = \Pr[\hat{R}f(\mathbf{x})]$. For the two-point correlator, we must have

$$\xi(\mathbf{x},\mathbf{y}) = \xi(\mathsf{R}^{-1}\mathbf{x},\mathsf{R}^{-1}\mathbf{y}) \quad \forall\mathsf{R}\,. \tag{2.2.5}$$

Combining statistical homogeneity and isotropy gives

$$\begin{aligned} &\xi(\mathbf{x},\mathbf{y}) = \xi\left(\mathsf{R}^{-1}(\mathbf{x}-\mathbf{y})\right) \quad \forall\mathsf{R} \\ \Rightarrow\quad &\xi(\mathbf{x},\mathbf{y}) = \xi(|\mathbf{x}-\mathbf{y}|)\,, \end{aligned} \tag{2.2.6}$$

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[1] For a recent review on the question of why quantum fluctuations from inflation can be treated as classical, see Keifer & Polarski (2008), available online at http://arxiv.org/abs/0810.0087.

so the two-point correlator depends only on the distance between the two points. Note that this holds even if correlating fields at different times, or correlating different fields.

We can repeat these arguments to constrain the form of the correlators in Fourier space. We adopt the symmetric Fourier convention, so that

$$f(\mathbf{k}) = \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \quad \text{and} \quad f(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} f(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} . \tag{2.2.7}$$

Note that for real fields, $f(\mathbf{k}) = f^*(-\mathbf{k})$. Under translations, the Fourier transform acquires a phase factor:

$$\begin{aligned} \hat{T}_{\mathbf{a}} f(\mathbf{k}) &= \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} f(\mathbf{x}-\mathbf{a}) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{d^3\mathbf{x}'}{(2\pi)^{3/2}} f(\mathbf{x}') e^{-i\mathbf{k}\cdot\mathbf{x}'} e^{-i\mathbf{k}\cdot\mathbf{a}} \\ &= f(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{a}} . \end{aligned} \tag{2.2.8}$$

Invariance of the two-point correlator in Fourier space is then

$$\begin{aligned} &\langle f(\mathbf{k}) f^*(\mathbf{k}')\rangle = \langle f(\mathbf{k}) f^*(\mathbf{k}')\rangle e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{a}} \quad \forall \mathbf{a} \\ \Rightarrow\quad &\langle f(\mathbf{k}) f^*(\mathbf{k}')\rangle = F(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}') , \end{aligned} \tag{2.2.9}$$

for some (real) function $F(\mathbf{k})$. We see that different Fourier modes are *uncorrelated.* Under rotations,

$$\begin{aligned} \hat{R} f(\mathbf{k}) &= \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} f(\mathsf{R}^{-1}\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{d^3\mathbf{x}}{(2\pi)^{3/2}} f(\mathsf{R}^{-1}\mathbf{x}) e^{-i(\mathsf{R}^{-1}\mathbf{k})\cdot(\mathsf{R}^{-1}\mathbf{x})} \\ &= f(\mathsf{R}^{-1}\mathbf{k}) , \end{aligned} \tag{2.2.10}$$

so, additionally demanding invariance of the two-point correlator under rotations implies

$$\langle \hat{R} f(\mathbf{k}) [\hat{R} f(\mathbf{k}')]^*\rangle = \langle f(\mathsf{R}^{-1}\mathbf{k}) f^*(\mathsf{R}^{-1}\mathbf{k}')\rangle = F(\mathsf{R}^{-1}\mathbf{k})\delta(\mathbf{k}-\mathbf{k}') = F(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}') \quad \forall \mathsf{R} . \tag{2.2.11}$$

(We have used $\delta(\mathsf{R}^{-1}\mathbf{k}) = \det\mathsf{R}\,\delta(\mathbf{k}) = \delta(\mathbf{k})$ here.) This is only possible if $F(\mathbf{k}) = F(k)$ where $k \equiv |\mathbf{k}|$. We can therefore define the *power spectrum*, $\mathcal{P}_f(k)$, of a homogeneous and isotropic field, $f(\mathbf{x})$, by

$$\langle f(\mathbf{k}) f^*(\mathbf{k}')\rangle = \frac{2\pi^2}{k^3} \mathcal{P}_f(k) \delta(\mathbf{k}-\mathbf{k}') . \tag{2.2.12}$$

The normalisation factor $2\pi^2/k^3$ in the definition of the power spectrum is conventional and has the virtue of making $\mathcal{P}_f(k)$ dimensionless if $f(\mathbf{x})$ is.

The correlation function is the Fourier transform of the power spectrum:

$$\begin{aligned} \langle f(\mathbf{x}) f(\mathbf{y})\rangle &= \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{d^3\mathbf{k}'}{(2\pi)^{3/2}} \underbrace{\langle f(\mathbf{k}) f^*(\mathbf{k}')\rangle}_{\frac{2\pi^2}{k^3}\mathcal{P}_f(k)\delta(\mathbf{k}-\mathbf{k}')} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k}'\cdot\mathbf{y}} \\ &= \frac{1}{4\pi} \int \frac{dk}{k} \mathcal{P}_f(k) \int d\Omega_{\mathbf{k}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} . \end{aligned} \tag{2.2.13}$$

We can evaluate the angular integral by taking $\mathbf{x}-\mathbf{y}$ along the z-axis in Fourier space. Setting $\mathbf{k}\cdot(\mathbf{x}-\mathbf{y}) = k|\mathbf{x}-\mathbf{y}|\mu$, the integral reduces to

$$2\pi \int_{-1}^{1} d\mu\, e^{ik|\mathbf{x}-\mathbf{y}|\mu} = 4\pi j_0(k|\mathbf{x}-\mathbf{y}|) , \tag{2.2.14}$$

where $j_0(x) = \sin(x)/x$ is a spherical Bessel function of order zero. It follows that

$$\xi(\mathbf{x},\mathbf{y}) = \int \frac{dk}{k}\, \mathcal{P}_f(k) j_0(k|\mathbf{x}-\mathbf{y}|) . \tag{2.2.15}$$

Note that this only depends on $|\mathbf{x}-\mathbf{y}|$ as required by Eq. (2.2.6).

The variance of the field is $\xi(0) = \int d\ln k\, \mathcal{P}_f(k)$. A *scale-invariant* spectrum has $\mathcal{P}(k) =$ const. and its variance receives equal contributions from every decade in k.

B. Gaussian random fields

For a Gaussian (homogeneous and isotropic) random field, $\Pr[f(\mathbf{x})]$ is a Gaussian functional of $f(\mathbf{x})$. If we think of discretising the field in N pixels, so it is represented by a N-dimensional vector $\mathbf{f} = [f(\mathbf{x}_1), f(\mathbf{x}_2), \ldots, f(\mathbf{x}_N)]^T$, the probability density function for $\mathbf{f}$ is a multi-variate Gaussian fully specified by the correlation function

$$\langle f_i f_j \rangle = \xi(|\mathbf{x}_i - \mathbf{x}_j|) \equiv \xi_{ij} \, , \tag{2.2.16}$$

where $f_i \equiv f(\mathbf{x}_i)$, so that

$$\Pr(\mathbf{f}) \propto \frac{e^{-f_i \xi_{ij}^{-1} f_j}}{\sqrt{\det(\xi_{ij})}} \, . \tag{2.2.17}$$

Since $f(\mathbf{k})$ is linear in $f(\mathbf{x})$, the probability distribution for $f(\mathbf{k})$ is also a multi-variate Gaussian. Since different Fourier modes are uncorrelated (see Eq. 2.2.9), they are statistically *independent* for Gaussian fields.

Inflation predicts fluctuations that are very nearly Gaussian and this property is preserved by *linear* evolution. The cosmic microwave background probes fluctuations mostly in the linear regime and so the fluctuations look very Gaussian (see Fig. 2). Non-linear structure formation at late times destroys Gaussianity and gives the filamentary cosmic web (see Fig. 1). Searching for primordial non-Gaussianity to probe departures from simple inflation is a very hot topic but no convincing evidence for primordial non-Gaussianity has yet been found.

C. Random fields on the sphere

Spherical harmonics form a basis for (square-integrable) functions on the sphere:

$$f(\hat{\mathbf{n}}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} f_{lm} Y_{lm}(\hat{\mathbf{n}}) \, . \tag{2.2.18}$$

The Y_{lm} are familiar from quantum mechanics as the position-space representation of the eigenstates of $\hat{L}^2 = -\nabla^2$ and $\hat{L}_z = -i\partial_\phi$:

$$\begin{aligned} \nabla^2 Y_{lm} &= -l(l+1) Y_{lm} \\ \partial_\phi Y_{lm} &= im Y_{lm} \, , \end{aligned} \tag{2.2.19}$$

with l an integer ≥ 0 and m an integer with $|m| \leq l$. The spherical harmonics are orthonormal over the sphere,

$$\int d\hat{\mathbf{n}} \, Y_{lm}(\hat{\mathbf{n}}) Y^*_{l'm'}(\hat{\mathbf{n}}) = \delta_{ll'} \delta_{mm'} \, , \tag{2.2.20}$$

so that the *spherical multipole coefficients* of $f(\hat{\mathbf{n}})$ are

$$f_{lm} = \int d\hat{\mathbf{n}} \, f(\hat{\mathbf{n}}) Y^*_{lm}(\hat{\mathbf{n}}) \, . \tag{2.2.21}$$

There are various phase conventions for the Y_{lm}; here we adopt $Y^*_{lm} = (-1)^m Y_{l\,-m}$ so that $f^*_{lm} = (-1)^m f_{l\,-m}$ for a real field.

What is the implication of statistical isotropy for the correlators of f_{lm}? For the two-point correlator, it turns out that we must have[2]

$$\langle f_{lm} f^*_{l'm'} \rangle = C_l \delta_{ll'} \delta_{mm'} \, , \tag{2.2.22}$$

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[2] A plausibility argument is as follows. Under rotations, the subset of the Y_{lm} with a given l (so $2l+1$ elements) transforms irreducibly so the $\delta_{ll'}$ form of the correlator is preserved under rotation. For rotation through γ about the z-axis,

$$Y_{lm}(\theta, \phi) \to Y_{lm}(\theta, \phi - \gamma) = e^{-im\gamma} Y_{lm}(\theta, \phi) \qquad \Rightarrow \qquad f_{lm} \to e^{-im\gamma} f_{lm} \, .$$

Under rotations,

$$\langle f_{lm} f^*_{l'm'} \rangle \to e^{-im\gamma} e^{im'\gamma} \langle f_{lm} f^*_{l'm'} \rangle \, ,$$

so invariance requires the correlator be $\propto \delta_{mm'}$.

where C_l is the *angular power spectrum* of f. What does this imply for the two-point correlation function? We have

$$\begin{aligned}\langle f(\hat{\mathbf{n}})f(\hat{\mathbf{n}}')\rangle &= \sum_{lm}\sum_{l'm'}\underbrace{\langle f_{lm}f^*_{l'm'}\rangle}_{C_l\delta_{ll'}\delta_{mm'}}Y_{lm}(\hat{\mathbf{n}})Y^*_{l'm'}(\hat{\mathbf{n}}') \\ &= \sum_l C_l \underbrace{\sum_m Y_{lm}(\hat{\mathbf{n}})Y^*_{lm}(\hat{\mathbf{n}}')}_{\frac{2l+1}{4\pi}P_l(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}')} = C(\theta)\,, \end{aligned} \tag{2.2.23}$$

where $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}' = \cos\theta$ and we used the addition theorem for spherical harmonics to express the sum of products of the Y_{lm} in terms of the Legendre polynomials $P_l(x)$. It follows that the two-point correlation function depends only on the angle between the two points, as required by statistical isotropy. Note that the variance of the field is

$$C(0) = \sum_l \frac{2l+1}{4\pi} C_l \approx \int d\ln l\, \frac{l(l+1)C_l}{2\pi}\,. \tag{2.2.24}$$

It is conventional to plot $l(l+1)C_l/(2\pi)$ which we see is the contribution to the variance per log range in l. Finally, we note that we can invert the correlation function to get the power spectrum by using orthogonality of the Legendre polynomials:

$$C_l = 2\pi \int_{-1}^{1} d\cos\theta\, C(\theta) P_l(\cos\theta)\,. \tag{2.2.25}$$

III. NEWTONIAN STRUCTURE FORMATION

Newtonian gravity is an adequate approximation of general relativity in cosmology on scales well inside the Hubble radius and when describing non-relativistic matter (for which the pressure P is much less than the energy density ρ). Newtonian gravity underlies all cosmological N-body simulations of the non-linear growth of structure and is much more intuitive than the full linearised treatment of general relativity (to be introduced later). In particular, in cosmology we can use the Newtonian treatment to describe sub-Hubble fluctuations in the cold dark matter (CDM) and baryons after decoupling.

Consider an ideal, self-gravitating non-relativistic fluid with density (for this section only, the *mass* density which, given our assumptions is essentially the total energy density) ρ, pressure $P \ll \rho$ and velocity $\mathbf{u}$. Denote the usual Newtonian position vector by $\mathbf{r}$ and time by t. The equations of motion of the fluid are as follows:

$$\text{Continuity} \qquad \partial_t \rho + \boldsymbol{\nabla}_{\mathbf{r}} \cdot (\rho \mathbf{u}) = 0 \tag{2.3.1}$$

$$\text{Euler} \qquad \partial_t \mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla}_{\mathbf{r}} \mathbf{u} = -\frac{1}{\rho} \boldsymbol{\nabla}_{\mathbf{r}} P - \boldsymbol{\nabla}_{\mathbf{r}} \Phi \tag{2.3.2}$$

$$\text{Poisson} \qquad \boldsymbol{\nabla}_{\mathbf{r}}^2 \Phi = 4\pi G \rho \,, \tag{2.3.3}$$

where the gravitational potential Φ determines the gravitational acceleration by $\mathbf{g} = -\boldsymbol{\nabla}_{\mathbf{r}} \Phi$. We can fudge the Poisson equation to get the correct Friedmann equations (see later) including the cosmological constant Λ by taking

$$\boldsymbol{\nabla}_{\mathbf{r}}^2 \Phi = 4\pi G \rho - \Lambda \,. \tag{2.3.4}$$

A. Background cosmology

To recover the background dynamics (described by the Friedmann equations), we consider a uniform expanding ball of fluid satisfying Hubble's law $\mathbf{u} = H(t)\mathbf{r}$. (Note the velocity goes to the speed of light at the Hubble radius!) This was covered in the third year cosmology course, but we include it again here from the perspective of the fluid equations.

Taking $\Phi = 0$ at $\mathbf{r} = 0$, the Poisson equation (2.3.4) integrates as

$$\begin{aligned} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) &= (4\pi G\rho - \Lambda) r^2 \\ \Rightarrow \qquad \frac{\partial \Phi}{\partial r} &= \frac{1}{3}(4\pi G \rho - \Lambda) r \\ \Rightarrow \qquad \Phi &= \frac{1}{6}(4\pi G\rho - \Lambda) r^2 \,. \end{aligned} \tag{2.3.5}$$

The Euler equation then becomes

$$\begin{aligned} \frac{\partial H}{\partial t}\mathbf{r} + H^2 \underbrace{\mathbf{r} \cdot \boldsymbol{\nabla}_{\mathbf{r}} \mathbf{r}}_{\mathbf{r}} &= -\frac{1}{3}(4\pi G\rho - \Lambda)\mathbf{r} \\ \Rightarrow \qquad \frac{\partial H}{\partial t} + H^2 &= \frac{1}{3}(\Lambda - 4\pi G \rho) \,. \end{aligned} \tag{2.3.6}$$

This is the Newtonian limit of one of the Friedmann equations (the relativistic result replaces ρ with the sum of the energy density and three times the pressure, $\rho + 3P$).

The continuity equation becomes

$$\begin{aligned} \partial_t \rho + \boldsymbol{\nabla}_{\mathbf{r}} \cdot [\rho(t) H(t) \mathbf{r}] &= 0 \\ \Rightarrow \qquad \partial_t \rho + 3\rho H &= 0 \,. \end{aligned} \tag{2.3.7}$$

This is the usual Friedmann statement of energy conservation for $\rho \ll P$. Introducing the scale factor a via $\partial_t a / a = H$, we have

$$\frac{1}{\rho}\frac{\partial \rho}{\partial t} + \frac{3}{a}\frac{\partial a}{\partial t} = 0 \quad \Rightarrow \quad \rho \propto a^{-3} \,, \tag{2.3.8}$$

which describes the dilution of the mass density by expansion.

Equations (2.3.6) and (2.3.7) have a first integral

$$-K = a^2 \left(H^2 - \frac{8\pi G}{3}\rho - \frac{1}{3}\Lambda \right) . \tag{2.3.9}$$

This is easily checked by differentiating:

$$\begin{aligned} -\frac{\partial K}{\partial t} &= 2a^2 H \left(H^2 - \frac{8\pi G}{3}\rho - \frac{1}{3}\Lambda \right) + a^2 \left(2H\frac{\partial H}{\partial t} - \frac{8\pi G}{3}\frac{\partial \rho}{\partial t} \right) \\ &= a^2 \left[2H^3 - \frac{16\pi G}{3} H\rho - \frac{2}{3}H\Lambda + 2H\left(-H^2 - \frac{4\pi G}{3}\rho + \frac{1}{3}\Lambda \right) + 8\pi G H \rho \right] \\ &= 0 \, . \end{aligned} \tag{2.3.10}$$

It follows that

$$H^2 + \frac{K}{a^2} = \frac{1}{3}(8\pi G\rho + \Lambda) \, . \tag{2.3.11}$$

In general relativity, K/a^2 is 1/6 of the intrinsic curvature of the surfaces of homogeneity.

B. Comoving coordinates

A comoving observer in the background (i.e. unperturbed) cosmology has velocity $d\mathbf{r}/dt = H(t)\mathbf{r}$ hence position $\mathbf{r} = a(t)\mathbf{x}$ where $\mathbf{x}$ is a constant. Rather than labelling events by t and $\mathbf{r}$, it is convenient to use t and $\mathbf{x}$, where $\mathbf{x}$ are *comoving spatial coordinates*: $\mathbf{x} = \mathbf{r}/a(t)$. Note these are *Lagrangian coordinates* in the background but not in the perturbed model.

Derivatives transform as follows:

$$\begin{aligned} \left(\frac{\partial}{\partial t}\right)_{\mathbf{r}} &= \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} + \left(\frac{\partial \mathbf{x}}{\partial t}\right)_{\mathbf{r}} \cdot \boldsymbol{\nabla}_{\mathbf{x}} \\ &= \left(\frac{\partial}{\partial t}\right)_{\mathbf{x}} - H(t)\mathbf{x}\cdot\boldsymbol{\nabla} \, , \end{aligned} \tag{2.3.12}$$

where we use $\boldsymbol{\nabla}$ to denote the gradient with respect to $\mathbf{x}$ at fixed t; and

$$\boldsymbol{\nabla}_{\mathbf{r}} = a^{-1}\boldsymbol{\nabla} \, . \tag{2.3.13}$$

In what follows, ∂_t should be understood as being taken at fixed $\mathbf{x}$.

C. Perturbation analysis

We now perturb ρ, $\mathbf{u}$ and Φ about their background values:

$$\rho \to \bar{\rho}(t) + \delta\rho \equiv \bar{\rho}(t)(1+\delta) \tag{2.3.14}$$

$$P \to \bar{P}(t) + \delta P \tag{2.3.15}$$

$$\mathbf{u} \to a(t)H(t)\mathbf{x} + \mathbf{v} \tag{2.3.16}$$

$$\Phi \to \bar{\Phi}(\mathbf{x}, t) + \phi \, . \tag{2.3.17}$$

Here, δ is the *fractional overdensity* in the fluid and ϕ the perturbed gravitational potential. Since, for a particle in the fluid,

$$\frac{d\mathbf{r}}{dt} = \frac{d(a\mathbf{x})}{dt} = aH\mathbf{x} + a\frac{d\mathbf{x}}{dt} = \mathbf{u} \, , \tag{2.3.18}$$

we see that $ad\mathbf{x}/dt = \mathbf{v}$, so the *peculiar velocity* $\mathbf{v}$ describes changes in the comoving coordinates of fluid elements in time (i.e. departures from the background Hubble flow).

The continuity equation (2.3.1) becomes (on using Eq. 2.3.12)

$$(1+\delta)\partial_t\bar{\rho} - H\bar{\rho}\mathbf{x}\cdot\boldsymbol{\nabla}\delta + \bar{\rho}\partial_t\delta + \frac{\bar{\rho}}{a}\boldsymbol{\nabla}\cdot[(1+\delta)(aH\mathbf{x}+\mathbf{v})] = 0\,. \tag{2.3.19}$$

Gathering terms that are zeroth, first and second-order in products of perturbed quantities gives

$$\underbrace{\partial_t\bar{\rho} + 3\bar{\rho}H}_{\text{0th}-\text{order}} + \underbrace{(\partial_t\bar{\rho} + 3\bar{\rho}H)\delta + \bar{\rho}\partial_t\delta + \frac{\bar{\rho}}{a}\boldsymbol{\nabla}\cdot\mathbf{v}}_{\text{1st}-\text{order}} + \underbrace{\frac{\bar{\rho}}{a}(\mathbf{v}\cdot\boldsymbol{\nabla}\delta + \delta\boldsymbol{\nabla}\cdot\mathbf{v})}_{\text{2nd}-\text{order}} = 0\,. \tag{2.3.20}$$

The background equation (2.3.7) sets the zero-order part to zero. In *linear perturbation theory*, we assume the perturbations are small enough (and their spatial derivatives) that we can ignore the second-order part, so that

$$\partial_t\delta + \frac{1}{a}\boldsymbol{\nabla}\cdot\mathbf{v} = 0\,. \tag{2.3.21}$$

`EXERCISE:` Show that Eqs (2.3.2) and (2.3.3) linearise to

$$\partial_t\mathbf{v} + H\mathbf{v} = -\frac{1}{a\bar{\rho}}\boldsymbol{\nabla}\delta P - \frac{1}{a}\boldsymbol{\nabla}\phi \tag{2.3.22}$$

$$\boldsymbol{\nabla}^2\phi = 4\pi G a^2\bar{\rho}\delta\,. \tag{2.3.23}$$

Scalar/vector decomposition

We can always decompose the vector $\mathbf{v}$ as

$$\mathbf{v} = \underbrace{\boldsymbol{\nabla}v}_{\text{scalar part}} + \underbrace{\mathbf{v}_\perp}_{\text{vector part}}\,, \tag{2.3.24}$$

where $\boldsymbol{\nabla}\cdot\mathbf{v}_\perp = 0$. It follows from Eq. (2.3.21) that the vector part of $\mathbf{v}$ does not lead to clumping of the matter. Since $\boldsymbol{\nabla}\times\mathbf{v} = \boldsymbol{\nabla}\times\mathbf{v}_\perp$, $\mathbf{v}_\perp$ describes the vorticity of the fluid – recalling that $\boldsymbol{\nabla}_\mathbf{r} = a^{-1}\boldsymbol{\nabla}$, the physical vorticity $\boldsymbol{\nabla}_\mathbf{r}\times\mathbf{u} = a^{-1}\boldsymbol{\nabla}\times\mathbf{v}_\perp$. In linear theory, the scalar and vector parts decouple. For example, consider the (comoving) curl of the perturbed Euler equation (2.3.22),

$$\boldsymbol{\nabla}\times\partial_t\mathbf{v} = \partial_t(\boldsymbol{\nabla}\times\mathbf{v}_\perp) = -H\boldsymbol{\nabla}\times\mathbf{v}_\perp\,. \tag{2.3.25}$$

It follows that $\boldsymbol{\nabla}\times\mathbf{v}_\perp$ decays as $1/a$ in an expanding universe so the vorticity falls as $1/a^2$. This decay of the vorticity is consistent with the circulation theorem, $\oint\mathbf{u}\cdot d\mathbf{r} = \text{const.}$ for a path comoving with the fluid. For general initial conditions, the peculiar velocity approaches a gradient at late times and the vector modes can be neglected. For initial conditions from inflation, vector modes are not excited in the first place. They are, however, important in models with continual sourcing of perturbations by cosmic defects.

D. Jeans' length

The time derivative of the perturbed continuity equation (2.3.21) gives

$$\partial_t^2\delta - \frac{1}{a}H\boldsymbol{\nabla}\cdot\mathbf{v} + \frac{1}{a}\boldsymbol{\nabla}\cdot\partial_t\mathbf{v} = 0. \tag{2.3.26}$$

Combining with the perturbed Euler equation (2.3.22) and the Poisson equation (2.3.23), we find

$$\partial_t^2\delta - \frac{1}{a}H\boldsymbol{\nabla}\cdot\mathbf{v} - \frac{1}{a}\boldsymbol{\nabla}\cdot\left(H\mathbf{v} + \frac{1}{a\bar{\rho}}\boldsymbol{\nabla}\delta P + \frac{1}{a}\boldsymbol{\nabla}\phi\right) = 0$$

$$\Rightarrow\quad \partial_t^2\delta - \frac{2}{a}H\boldsymbol{\nabla}\cdot\mathbf{v} - \frac{1}{a^2\bar{\rho}}\boldsymbol{\nabla}^2\delta P - \frac{1}{a^2}\boldsymbol{\nabla}^2\phi = 0$$

$$\Rightarrow\quad \partial_t^2\delta + 2H\partial_t\delta - 4\pi G\bar{\rho}\delta - \frac{1}{a^2\bar{\rho}}\boldsymbol{\nabla}^2\delta P = 0\,. \tag{2.3.27}$$

This is the fundamental equation for the growth of structure in Newtonian theory. It shows the general competition between infall by gravitational attraction – the $4\pi G\bar{\rho}\delta$ term – and pressure support, $\nabla^2\delta P$.

Consider a *barotropic* fluid such that $P = P(\rho)$; then

$$\delta P = \frac{\partial P}{\partial \rho}\bar{\rho}\delta \equiv c_s^2\bar{\rho}\delta \qquad (2.3.28)$$

where c_s^2 is the sound speed. Using this in Eq. (2.3.27), and Fourier expanding so that $\nabla^2 \to -k^2$, gives

$$\partial_t^2\delta + 2H\partial_t\delta + \left(\frac{c_s^2k^2}{a^2} - 4\pi G\bar{\rho}\right)\delta = 0\,. \qquad (2.3.29)$$

This is the equation for a damped (in an expanding universe) oscillator provided that

$$\frac{c_s^2k^2}{a^2} > 4\pi G\bar{\rho}\,, \qquad (2.3.30)$$

and, in this case, the pressure support gives rise to acoustic oscillations (sound waves) in the fluid. However, for $c_s^2k^2/a^2 < 4\pi G\bar{\rho}$, the system is unstable to gravitational accretion. Perturbations with *proper wavelength* $2\pi a/k$ exceeding the (proper) *Jeans' wavelength*,

$$\lambda_{\rm J} \equiv c_s\sqrt{\frac{\pi}{G\bar{\rho}}}\,, \qquad (2.3.31)$$

are gravitationally unstable, while on smaller scales pressure supports oscillations.

For a fluid with equation of state $P/\rho > -1/3$, the proper Jeans' length in an expanding universe grows faster than a comoving scale ($\propto a$). This has the consequence that Fourier modes of the perturbations that start off outside the Jeans' length, where they evolve by gravitational accretion, later come inside the Jeans' length and subsequently undergo acoustic oscillations.

The Jeans' length is roughly the radius R of a region of background density $\bar{\rho}$ such that the free-fall time, $t_{\rm ff}$, equals the sound-crossing time, $t_{\rm sound}$. To see this, note that the free-fall time is the time to collapse under gravity. Consider a shell of matter on the edge of a collapsing mass M of intitial radius R, such that at time t later the shell is at radius r. The mass enclosed is always M so that the gravitational acceleration is $-GM/r^2$. Equating this to $\partial_t^2 r$, and solving for a particle initially at rest, the shell collapses to $r = 0$ in the free-fall time where

$$t_{\rm ff} \sim \frac{R^{3/2}}{\sqrt{GM}} \sim \frac{1}{\sqrt{G\bar{\rho}}}\,. \qquad (2.3.32)$$

The sound-crossing time is simply $t_{\rm sound} = R/c_s$ and this equals the free-fall time for $R \sim c_s/\sqrt{G\bar{\rho}} \sim \lambda_{\rm J}$. Fluctuations larger than the Jeans' length do not have time for pressure to resist gravitational infall since the time to infall is less than the time it takes to propagate a pressure disturbance (i.e. a sound wave) across the perturbation. Note, finally, that the free-fall time is roughly the Hubble time, $1/H$, when curvature and dark energy are negligible.

E. Applications to cold dark matter

1. Solutions in an Einstein-de Sitter phase

After matter-radiation equality, but well before dark energy comes to dominate, our universe is well described by an Einstein-de Sitter model having $\bar{P} \approx 0$ and zero curvature or Λ. Scales of cosmological interest are much larger than the Jeans' scale for the baryons and so both CDM fluctuations and those for the baryons have the same dynamical equations. We shall show shortly that quickly after recombination, the fractional overdensity in the baryons, δ_b, approaches that in the CDM, δ_c, and the matter behaves like a single pressure-free fluid with total density contrast

$$\delta_m = \frac{\bar{\rho}_b\delta_b + \bar{\rho}_c\delta_c}{\bar{\rho}_b + \bar{\rho}_c} \approx \delta_c\,. \qquad (2.3.33)$$

Since $H^2 \propto \bar{\rho} \propto a^{-3}$, we have $a \propto t^{2/3}$ and so $H = 2/(3t)$ and $4\pi G\bar{\rho} = 2/(3t^2)$. Equation (2.3.27) then gives the evolution of the density fluctuations in the pressure-free matter as

$$\partial_t^2\delta_m + \frac{4}{3t}\partial_t\delta_m - \frac{2}{3t^2}\delta_m = 0\,. \qquad (2.3.34)$$

Trying solutions like t^p gives independent solutions $\delta_m \propto t^{-1}$ and $\delta_m \propto t^{2/3} \propto a$. The *growing-mode* solution of the density contrast therefore grows like the scale factor. Note that here, in an expanding universe, gravitational attraction has given rise to power-law growth of δ to be compared to the exponential growth predicted in a non-expanding model. The Poisson equation (2.3.23) tells us that the gravitational potential is constant since, in Fourier space,

$$-k^2\phi = 4\pi G a^2 \underbrace{\bar{\rho}}_{\propto a^{-3}} \underbrace{\delta}_{\propto a} = \text{const.} \tag{2.3.35}$$

2. The Meszaros effect

The Meszaros effect describes the way that CDM grows only logarithmically on scales inside the sound horizon during radiation domination. Generally, CDM (or anything else) feels the gravity of all clustered components so Eq. (2.3.27) generalises to the ith component of a set of non-interacting (except through gravity) fluids as

$$\partial_t^2 \delta_i + 2H\partial_t \delta_i - 4\pi G \sum_j \bar{\rho}_j \delta_j - \frac{1}{a^2 \bar{\rho}_i} \nabla^2 \delta P_i = 0\,. \tag{2.3.36}$$

Specialising to pressure-free CDM we have

$$\partial_t^2 \delta_c + 2H\partial_t \delta_c - 4\pi G \sum_j \bar{\rho}_j \delta_j = 0\,. \tag{2.3.37}$$

Our Newtonian treatment at least makes it plausible that the Jeans' length for perturbations in the radiation fluid (for which $c_s = 1/\sqrt{3}$) during radiation domination is of the order of the Hubble radius. Radiation fluctuations on scales smaller than this therefore oscillate as sound waves and their time-averaged density contrast vanishes (we shall show this properly when we develop relativistic perturbation theory). It follows that the CDM is essentially the only clustered component during the acoustic oscillations of the radiation, and so

$$\partial_t^2 \delta_c + \frac{1}{t}\partial_t \delta_c - 4\pi G \bar{\rho}_c \delta_c = 0\,, \tag{2.3.38}$$

where we used $a \propto t^{1/2}$ and so $H = 1/(2t)$. Since δ_c evolves only on cosmological timescales (it has no pressure support for it to do otherwise),

$$\partial_t^2 \delta_c \sim H^2 \delta_c \gg 4\pi G \bar{\rho}_c \delta_c \tag{2.3.39}$$

during radiation domination, as $\bar{\rho}_r \gg \bar{\rho}_c$. We can therefore ignore the last term in Eq. (2.3.38) compared to the others and we have solutions with $\delta_c = \text{const.}$ and $\delta_c \propto \ln t$. We see that the rapid expansion due to the effectively unclustered radiation reduces the growth of δ_c to only logarithmic.

3. Late-time suppression of structure formation by Λ

At late times, the dominant clustered component is the matter and we have

$$\partial_t^2 \delta_m + 2H\partial_t \delta_m - 4\pi G \bar{\rho}_m \delta_m = 0\,. \tag{2.3.40}$$

In matter domination, this reduces to Eq. (2.3.34) and δ_m grows like a, but when Λ comes to dominate $a \propto e^{t\sqrt{\Lambda/3}}$ and $H \approx \text{const.}$ It follows that $4\pi G\bar{\rho}_m \ll H^2$ (currently $4\pi G\bar{\rho}_m/H^2 \sim 0.37$) and

$$\partial_t^2 \delta_m + 2H\partial_t \delta_m \approx 0\,. \tag{2.3.41}$$

The solutions of this are $\delta_m = \text{const.}$ or $\delta_m \propto e^{-2t\sqrt{\Lambda/3}} \propto a^{-2}$ and Λ suppresses the growth of structure. Note also that a constant density contrast implies that the gravitational potential decays as $a^2\bar{\rho}_m \propto a^{-1}$. This leaves an imprint in the CMB called the *integrated Sachs-Wolfe effect.*

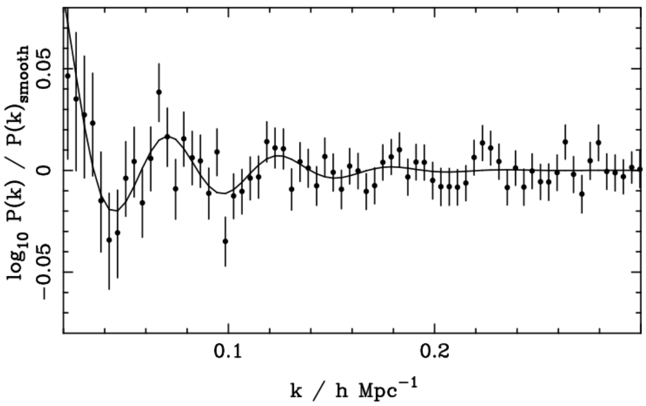

FIG. 3 Ratio of the matter power spectrum to a smooth spectrum (i.e. a model with no baryons) showing the expected baryon acoustic oscillations. Credit: Percival et al.

4. Evolution of baryon fluctuations after decoupling

Before decoupling, the baryon dynamics is linked to that of the radiation by efficient (Compton) scattering. On sub-Hubble scales, δ_b oscillates like the radiation but, after matter-radiation equality, δ_c grows like a. It follows that just after decoupling, $\delta_c \gg \delta_b$. Subsequently, the baryons fall into the potential wells sourced mainly by the CDM and $\delta_b \to \delta_c$ as we shall now show.

Ignoring baryon pressure and Λ, the coupled dynamics of the baryon and CDM fluids after decoupling is approximately given by

$$\partial_t^2 \delta_b + \frac{4}{3t}\partial_t \delta_b = 4\pi G(\bar{\rho}_b \delta_b + \bar{\rho}_c \delta_c) \tag{2.3.42}$$

$$\partial_t^2 \delta_c + \frac{4}{3t}\partial_t \delta_c = 4\pi G(\bar{\rho}_b \delta_b + \bar{\rho}_c \delta_c)\,. \tag{2.3.43}$$

We can decouple these equations by using normal coordinates δ_m (see equation 2.3.33) and $\Delta \equiv \delta_c - \delta_b$. Then

$$\partial_t^2 \Delta + \frac{4}{3t}\partial_t \Delta = 0 \quad \Rightarrow \quad \Delta = \text{const. or } \Delta \propto t^{-1/3}\,, \tag{2.3.44}$$

while δ_m follows Eq. (2.3.34) and has solutions $\propto t^{-1}$ and $t^{2/3}$. Since

$$\frac{\delta_c}{\delta_b} = \frac{\bar{\rho}_m \delta_m + \bar{\rho}_b \Delta}{\bar{\rho}_m \delta_m - \bar{\rho}_c \Delta} \to \frac{\delta_m}{\delta_m} = 1\,, \tag{2.3.45}$$

we see that δ_b approaches δ_c.

The non-zero initial value of δ_b at decoupling, and, more importantly $\partial_t \delta_b$, leaves a small imprint in the late-time δ_m that oscillates with scale. These *baryon acoustic oscillations* have recently been detected in the clustering of galaxies (see Fig. 3).

IV. RELATIVISTIC STRUCTURE FORMATION

This section will be done entirely on the whiteboard.

V. INFLATION AND THE ORIGIN OF STRUCTURE

A. Schematic overview of origin of structure in the inflationary paradigm

So far, we have not discussed the origin of the primordial perturbation which provided the seeds for cosmological structure formation under the action of gravitational instability. The study of this question has the potential to expose deep connections between cosmology and physics at immensely high energies which are forever beyond the reach of earth-bound particle accelerators.

Besides solving the big bang puzzles, the decreasing comoving horizon during inflation is the key feature required for the quantum generation of cosmological perturbations. During inflation, quantum fluctuations are generated on sub-Hubble scales and are then stretched out of the Hubble radius by the accelerated expansion. In other words, the superluminal expansion stretches the perturbations to apparently acausal distances. They become classical superhorizon density perturbations which re-enter the Hubble radius in the subsequent non-accelerating evolution and then undergo gravitational collapse to form the large-scale structure in the universe.

We are most interested in the *scalar* perturbations to the metric, as these couple to the density of matter and radiation, and are ultimately responsible for most of the inhomogeneities and anisotropies in the universe. In addition to scalar perturbations, however, inflation also generates *tensor* fluctuations in the gravitational metric (called *gravitational waves*). These are not coupled to the density and thus are not responsible for the large-scale structure of the universe. However, they do induce fluctuations in the cosmic microwave background (CMB). In fact, these fluctuations turn out to be a unique signature of inflation and offer the best window on the physics driving inflation.

We will study tensor perturbations before scalar perturbations for reasons of simplicity. Tensor perturbations to the metric due to a scalar field are not coupled to any other perturbation variables, so when we consider them, we are looking at the fluctuations in a single field. Scalar perturbations to the metric couple to energy density fluctuations. The coupled fields fluctuate together and complicate the maths. This distracts from the main point, which is that quantum mechanical fluctuations during inflation are responsible for the perturbations around the smooth background that ultimately gives rise to all the structure in universe. So we will first introduce this idea in the simplified context of a single field, and start with tensor perturbations.

During inflation, the universe consists primarily of a uniform scalar field and a uniform background metric. Against this background, the fields fluctuate quantum mechanically. Perturbations of the inflaton field value $\delta\phi$ satisfy the equation of motion of a harmonic oscillator with time-dependent mass. The quantum treatment of inflaton perturbations therefore parallels the quantum treatment of a collection of one-dimensional harmonic oscillators. Just as zero-point fluctuations of a harmonic oscillator induce a non-zero variance for the oscillation amplitude $\langle x^2 \rangle$, the quantum fluctuations of a light scalar field[3] during inflation induce a non-zero variance for the inflaton perturbations. Our goal is to compute this variance and see how it evolves as inflation progresses. Once we know this variance, we can draw from a distribution with this variance to set the initial conditions.

B. Quantizing the harmonic oscillator

In order to compute the quantum fluctuations in the metric, we need to quantize the field. For both scalar and tensor perturbations, the easiest way to do this is to rewrite the problem so that it looks like a simple harmonic oscillator (SHO).

Let us remind ourselves a few facts about the quantization of an SHO.

- An SHO with unit mass and frequency ω obeys

$$\boxed{\ddot{x} + \omega^2 x = 0}\,. \tag{2.5.1}$$

- Upon quantitization, x becomes a quantum operator

$$\hat{x} = v(\omega, t)\hat{a} + v^*(\omega, t)\hat{a}^\dagger \tag{2.5.2}$$

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[3] Such a field has (effective) mass $V_{,\Phi\Phi} \ll H^2$ which is equivalent to $|\eta_V| \ll 1$.

where $\hat{a}$ is a quantum operator and $v \propto \exp(i\omega t)$ is a solution to the SHO equation (2.5.1).

- The operator $\hat{a}$ annihilates the vacuum state, $\hat{a}|0\rangle = 0$ (in which there are no particles) and satisfies the commutation relation

$$[\hat{a}, \hat{a}^\dagger] \equiv \hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a} = 1\,, \tag{2.5.3}$$

giving variance

$$\langle |\hat{x}|^2 \rangle \equiv \langle 0|\hat{x}^\dagger \hat{x}|0\rangle = |v(\omega, t)|^2. \tag{2.5.4}$$

`EXERCISE:` If you need to refresh your memory for quantization of the SHO, read Sec. 6.4.1. in Dodelson.

C. Tensor perturbations

Tensor perturbations are characterized by a metric with $g_{00} = 1$, zero space-time components $g_{0i} = 0$, and spatial elements

$$g_{ij} = -a^2 \begin{pmatrix} 1 + h_+ & h_\times & 0 \\ h_\times & 1 - h_+ & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.5.5}$$

That is, the perturbations to the metric are described by two functions h_+ and $h_\times$, assumed small. For definiteness, we have chosen the perturbations to be in the x–y plane. This corresponds to an implicit choice of axes; in particular it corresponds to choosing the z axis to be in the direction of the wavevector $\vec{k}$.

More generally, h_+ and $h_\times$ are two components of a *divergenceless*, *traceless*, *symmetric* tensor. If this perturbation tensor is written as $\mathcal{H}_{ij}$, *divergenceless* means that $k^i \mathcal{H}_{ij} = k^j \mathcal{H}_{ij} = 0$. This is clearly satisfied by (2.5.5) since there are no components in the $\hat{k} = \hat{z}$ direction. Tracelessness is also satisfied since the sum of perturbations along the diagonal vanishes.

Once the metric has been written down, we know what to do - the procedure is identical to what we used in Part I of the course to derive the Einstein equations for the unperturbed metric. The derivation proceeds as usual in three steps: (i) Christoffel symbols, (ii) Ricci tensor, and (iii) Ricci scalar.

`EXERCISE:` Work through the GR machinery with the aid of Sec. 5.3 of Dodelson. The minimum learning outcome of this exercise should be to understand every step of this derivation, even if you ultimately can't reproduce it with the book closed.

During inflation, gravitational waves are not sourced by the scalar field – i.e. perturbing the energy-momentum tensor of a scalar field leads to a zero RHS in the relevant perturbed Einstein equation (see Exercise 10 in Chapter 6 of Dodelson). The perturbed Einstein equation thus leads to

$$\boxed{h''_\alpha + 2\left(\frac{a'}{a}\right) h'_\alpha + k^2 h_\alpha = 0}\,, \tag{2.5.6}$$

where $\alpha = +, \times$ and primes denote derivatives with respect to conformal time η. Eq. (2.5.6) is a wave equation, and the corresponding solutions are called *gravitational waves*. For example, if we neglect the expansion of the universe so that the damping term in (2.5.6) vanishes, the equation is manifestly in the form of an SHO equation and we immediately see that the two solutions are $h_\alpha \propto e^{\pm ik\eta}$. In real space, the perturbation to the metric is in the form

$$h_\alpha(\vec{x}, \eta) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}} \left[A e^{ik\eta} + B e^{-ik\eta} \right] \quad \text{(no expansion)}\,. \tag{2.5.7}$$

These two modes correspond to waves travelling in the $\pm\hat{z}$ direction at the speed of light. Eq. (2.5.6) is a generalization of the wave equation to an expanding universe.

EXERCISE: Solve the wave equation (2.5.6) if the universe is purely matter-dominated. Do the same for the radiation-dominated case.

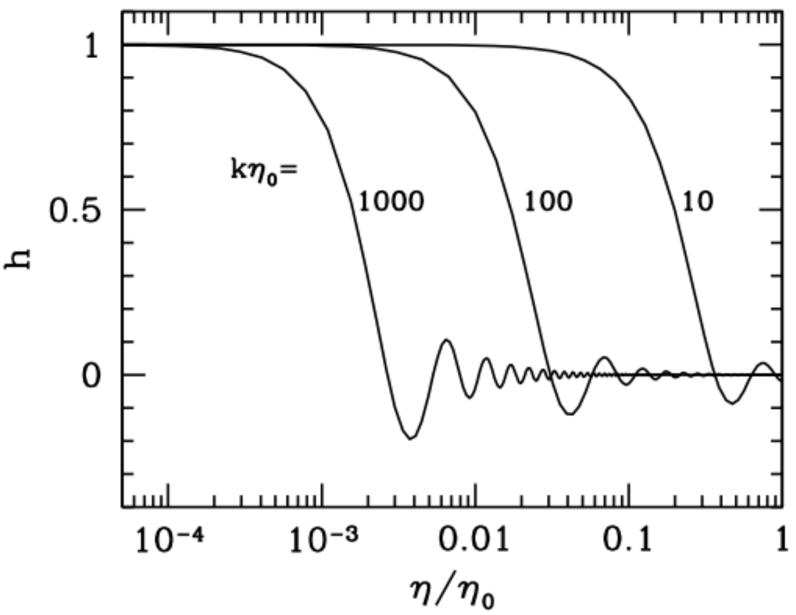

FIG. 4 Evolution of gravitational waves as a function of conformal time. Three different modes are shown, labelled by their wavenumbers. Smaller scale modes decay earlier. Figure credit: *Modern Cosmology*, Dodelson.

The solutions are oscillatory, like the simple ones in (2.5.7), but also damp out. Fig. 4 shows the evolution of h_α for three different wavelength modes. The large scale mode (with $k\eta_0 = 10$) remains constant at early times when its wavelength is larger than the horizon, $k\eta < 1$. Once its wavelength becomes comparable to the horizon, $k\eta \sim 1$, the solution oscillates several times until the present epoch and the amplitude begins to die off. The small scale mode $k\eta_0 = 1000$ illustrated here also begins to decay when its wavelength becomes comparable to the horizon. Its horizon entry occurs much earlier, though, so the decay is much more efficient, and by today, its amplitude is extremely small.

An important point about the effect of gravity waves on the CMB anisotropy spectrum can be gleaned from this discussion. Because small-scale modes decay earlier than large scale modes, at decoupling (at $\eta/\eta_0 \simeq 0.02$) only modes with $k\eta_0 \gtrsim 100$ persist. All smaller scale modes can be neglected. Therefore, anisotropies on small angular scales will not be affected by gravitational waves. Only the large-scale anisotropies are affected.

Now let us return to eq. (2.5.6). To continue with our programme of parsimoniously re-using the previous knowledge on quantizing the SHO, we would like to massage this equation into the form of a SHO, so that h can be easily quantized.

To do this, define

$$\tilde{h} \equiv \frac{ah}{\sqrt{16\pi G}}\,, \tag{2.5.8}$$

where we have cunningly normalized the field to match the canonical normalization of a scalar field[4]. Working through

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[4] If you are interested in this normalization factor, read the footnote at the bottom of pp158 in Dodelson.

the maths of substituting this redefinition into (2.5.6), we arrive at:

$$\tilde{h}'' + \left(k^2 - \frac{a''}{a}\right)\tilde{h} = 0\,. \tag{2.5.9}$$

`EXERCISE:` Complete the missing steps to change the variable in (2.5.6) to arrive at (2.5.9).

We have now arrived at the form of an SHO, which we know how to use! The equation now has no damping term ($\propto \tilde{h}'$) so we can immediately write down an expression for the quantum operator

$$\hat{\tilde{h}}(\vec{k},\eta) = v(k,\eta)\hat{a}_{\vec{k}} + v^*(k,\eta)a^\dagger_{\vec{k}}\,, \tag{2.5.10}$$

where the coefficients of the creation and annihilation operators satisfy the equation

$$v'' + \left(k^2 - \frac{a''}{a}\right)v = 0\,. \tag{2.5.11}$$

Further, we know the variance of the perturbations

$$\langle\hat{\tilde{h}}^\dagger(\vec{k},\eta)\hat{\tilde{h}}(\vec{k'},\eta)\rangle = |v(k,\eta)|^2\delta^3(\vec{k}-\vec{k'})\,. \tag{2.5.12}$$

After transforming back to the h field, we see that

$$\begin{aligned}\langle\hat{h}^\dagger(\vec{k},\eta)\hat{h}(\vec{k'},\eta)\rangle &= \frac{16\pi G}{a^2}|v(\vec{k},\eta)|^2\delta^3(\vec{k}-\vec{k'})\,, \\ &\equiv \frac{2\pi^2}{k^3}P_h(k)\delta^3(\vec{k}-\vec{k'})\end{aligned} \tag{2.5.13}$$

where the second line defines the *power spectrum* of the primordial perturbations to the metric. Note that conventions abound in the definition of the primordial power spectrum. We have followed the usual early universe community definition which gives a dimensionless power spectrum. Dodelson follows the convention of the large scale structure community where the power spectrum is taken to have dimensions of k^{-3}. Further, Dodelson is using a different Fourier convention to this class, and we use the one which is more conventional in the field. If you find this frustrating, you are not alone.

Thus, with this definition, we have

$$P_h(k) = 16\pi G\left(\frac{k^3}{2\pi^2}\right)\frac{|v(k,\eta)|^2}{a^2}\,. \tag{2.5.14}$$

To determine the spectrum of tensor perturbations produced during inflation, we have to now solve the second order differential equation (2.5.16) for $v(k,\eta)$.

`EXERCISE:` Show that, during inflation,

$$\left(\frac{a''}{a}\right) \simeq \frac{2}{\eta^2}\,. \tag{2.5.15}$$

.

The relevant equation therefore becomes

$$v'' + \left(k^2 - \frac{2}{\eta^2}\right)v = 0\,. \tag{2.5.16}$$

The initial conditions necessary to solve this equation come from considering v at very early times before inflation has done most of its work. At that time, $-\eta$ is large, of order $\eta_{\rm prim}$, so the k^2 term dominates, and the equation reduces precisely to that of the SHO. In that case, we know that the properly normalized solution is $e^{-ik\eta}/\sqrt{2k}$. The solution which correctly yields this limit is

$$v = \frac{e^{-ik\eta}}{\sqrt{2k}}\left[1 - \frac{i}{k\eta}\right] . \tag{2.5.17}$$

This obviously goes into the correct solution when the mode is well within the horizon ($k|\eta| \gg 1$).

`EXERCISE:` Check that (2.5.17) is a solution of (2.5.16).

The evolution of a mode h with wavenumber k according to this solution can be interpreted as follows. When the mode is well within the horizon ($k|\eta| \gg 1$), the amplitude of h decays as $h \propto 1/a$,

$$v \to \frac{e^{-ik\eta}}{\sqrt{2k}} \implies h \propto \frac{1}{a} \qquad\qquad k|\eta| \gg 1\, . \tag{2.5.18}$$

At $k\eta = 1$, the mode leaves the horizon. Well outside the horizon, ($-k\eta \to 0$), h becomes constant:

$$v \to \frac{e^{-ik\eta}}{\sqrt{2k}}\frac{i}{k\eta} \implies h \to \text{const} \qquad\qquad -k\eta \to 0\, . \tag{2.5.19}$$

The primordial power spectrum for tensor modes, which scales as $|v|^2/a^2$, is thus constant after the mode exits the horizon. This constant determines the initial conditions with which to start off $h_{\times,+}$ at early times. Here, "early" means well after inflation has ended but before decoupling. The primordial power spectrum for tensor modes is then

$$\begin{aligned} P_h(k) &= \frac{16\pi G}{a^2}\left(\frac{k^3}{2\pi^2}\right)\frac{1}{2k^3\eta^2} \\ &\simeq 16\pi G\left(\frac{H}{2\pi}\right)^2 . \end{aligned} \tag{2.5.20}$$

We have assumed H is constant in deriving the last line. Specifically, during slow-roll inflation the fractional variation in the Hubble rate per Hubble time is small so we can treat H as a constant, H_k, for the few e-folds either side of Hubble exit. Then, in this interval, $a = -(H_k\eta)^{-1}$. More generally, H has to be evaluated at the time when the mode of interest leaves the horizon. Since $H \sim$ constant during inflation, P_h is nearly scale-invariant. Our expression also implies that the detection of primordial gravitational waves measures the Hubble rate during inflation! Since the Hubble rate is dominated by potential energy during inflation, we would also measure $V(\phi)$ at horizon exit!

Further, the fluctuations in h are Gaussian, just like the quantum-mechanical fluctuations of the SHO. Gaussianity is a fairly robust prediction of inflation (and thus, a means of testing the inflationary picture observationally). Finally, Eq. (2.5.20) is the power spectrum for h_+ and $h_\times$ separately, and each polarization contributes twice to the metric perturbations; these are uncorrelated, so the power spectrum for all modes must be multiplied by a factor of 4. Thus, we finally obtain the tensor power spectrum for inflation:

$$\boxed{P_h = 64\pi G\left(\frac{H}{2\pi}\right)^2 \equiv \frac{8}{M_{\rm Pl}^2}\left(\frac{H}{2\pi}\right)^2} . \tag{2.5.21}$$

D. Scalar perturbations

A pedagogical treatment of the generation of density (scalar) perturbations is beyond the scope of this course and we direct the interested reader to the comprehensive treatment in Kinney[5] (2009). The quantity we wish to compute

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[5] http://arxiv.org/abs/0902.1529

is the comoving curvature perturbation $\mathcal{R}$, since this is conserved from Hubble exit during inflation to Hubble re-entry during the standard radiation or matter-dominated epochs. With $\mathcal{R}$ we can reliably compute the primordial fluctuation (in single-field models of inflation) in the radiation era without needing to model the dynamics of the reheating process.

Let's decompose the scalar field into a zero-order homogeneous part and a perturbation:

$$\phi(\vec{x},t) = \phi^{(0)}(t) + \delta\phi(\vec{x},t)\,, \tag{2.5.22}$$

and find an equation governing $\delta\phi$ assuming the smoothly expanding FRW metric. Under this assumption, only first order pieces are perturbations to $T_{\mu\nu}$, and after the usual Einstein equation manipulations, we find

$$\delta\phi'' + 2\frac{a'}{a}\delta\phi' + k^2\delta\phi = 0\,. \tag{2.5.23}$$

This equation has the same form as the tensor case (2.5.6), and we can trivially copy the solution, dropping the tensor normalization factor $16\pi G$ as we already have a canonical scalar field:

$$P_{\delta\phi} = \left(\frac{H}{2\pi}\right)^2\,. \tag{2.5.24}$$

The right-hand side is evaluated when the mode k exits the Hubble radius. Since H varies very slowly during inflation, the scalar power spectrum is also *nearly (but not exactly) scale-invariant*. This is one of the most generic and important observational predictions of inflation.

In order to sketch how fluctuations in ϕ get transferred into density fluctuations, we will work in the *Conformal Newtonian gauge* (CNG), where the perturbed FRW metric is:

$$\boxed{ds^2 = a^2(\eta)[(1+2\Psi)d\eta^2 - \delta_{ij}(1-2\Phi)dx^i dx^j]}\,. \tag{2.5.25}$$

This metric describes *scalar* metric perturbations[6] using the *scalar potentials* Ψ (Newtonian potential) and Φ (perturbation to the spatial curvature).

We want to know how fluctuations in ϕ get transferred to Ψ (or Φ, assumed identical in magnitude here[7]) so that we can relate inflationary perturbations to initial density fluctuations. We can then use the transfer functions derived in the previous section to evolve them forward in time to any epoch after inflation ends, and compare with observations.

Until now, we have neglected the metric perturbations. When the wavelength of the perturbation is of the order of the horizon or smaller, this approximation is valid. However, by the end of inflation, the metric perturbation becomes important. Although the inflation-induced perturbations start off as all-$\delta\phi$, they end up as a linear combination of Ψ and $\delta\phi$, or more generally, as a linear combination of Ψ and perturbations to the energy momentum tensor. The trick is to find the linear combination which is conserved outside the horizon. The value of this conserved linear combination is determined by $\delta\phi$ at horizon crossing. We can then evaluate it after inflation solely in terms of Ψ. Finding the linear relation between Ψ and $\delta\phi$ will immediately allow us to obtain P_Ψ in terms $P_{\delta\phi}$ obtained above.

The easiest way to do this is to switch to a gauge where the spatial part of the metric is unperturbed: a *spatially flat slicing*[8]. In such a gauge, the previous result (2.5.24) is exact. Now, the question is how to move back to the CNG. The answer is that we need to (i) find a *gauge invariant variable* $\propto \delta\phi$ in a spatially flat slicing (SFS), and (ii) find this variable in CNG, thus linking Φ in CNG with $\delta\phi$ in SFS.

`READING:` (**NON-EXAMINABLE**) You can see these steps fleshed out on §6.5.3 (pp 169) of Dodelson. Though it sounds gory, it is a compact and elegant calculation.

In the SFS, the line element (with A, B characterizing the line element) is given by

$$ds^2 = (1+2A)dt^2 + 2aB_{,i}dx^i dt - \delta_{ij}a^2 dx^i dx^j\,. \tag{2.5.26}$$

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[6] By the *Decomposition Theorem*, scalars, vectors and tensors evolve independently. Only scalars couple to matter.
[7] This assumption is equivalent to neglecting anisotropic stresses in the energy-momentum tensor.
[8] Technically, the field fluctuation here is defined on hypersurfaces with zero intrinsic curvature and, in this gauge, the result is the same as if metric perturbations (i.e. the back-reaction of $\delta\phi$ on the spacetime geometry) were ignored.

It turns out that the appropriate gauge-invariant combo that translates between the two gauges is the previously-heralded *comoving curvature perturbation*,

$$\boxed{\mathcal{R} = -\frac{aH}{\partial_\eta \phi^{(0)}}\delta\phi}\,, \tag{2.5.27}$$

immediately giving us the power spectrum,

$$\boxed{P_{\mathcal{R}} = \left(\frac{aH}{\partial_\eta \phi^{(0)}}\right)^2 P_{\delta\phi} \equiv \left(\frac{H}{\partial_t \phi^{(0)}}\right)^2 \left(\frac{H}{2\pi}\right)^2 \equiv \left(\frac{H^2}{2\pi\partial_t \phi^{(0)}}\right)^2}\,. \tag{2.5.28}$$

The right hand side is again evaluated at horizon exit, $k = aH$.

During inflation, comoving hypersurfaces have the property that they coincide with the hypersurfaces over which the (total) inflaton ϕ is homogeneous. These hypersurfaces are not the same as the zero-curvature surfaces on which Eq. (2.5.24) holds – there is a time delay between them, $\delta t = -\delta\phi/\partial_t\phi^{(0)}$, such that the evolution of the background $\phi^{(0)}$ in this time compensates for the perturbation $\delta\phi$ to give a smooth total ϕ. The differential background expansion during this time delay means that the intrinsic curvature of the comoving hypersurfaces is simply $\mathcal{R} = -H\delta\phi/\partial_t\phi(0)$. This is reason for the *curvature perturbation* designation.

E. Slow-roll expansion

In this section, to avoid cumbersome notation we set $\bar{\Phi} \equiv \phi^{(0)}$. For a slowly-rolling scalar field, we can express $\mathcal{P}_{\mathcal{R}}(k)$ directly in terms of the inflaton potential. During slow roll, the potential energy of the field dominates over the kinetic energy and so

$$H^2 \approx \frac{1}{3M_{\rm Pl}^2} V(\bar{\Phi})\,, \tag{2.5.29}$$

and the field evolution is friction limited:

$$3H\partial_t\bar{\Phi} = -V_{,\bar{\Phi}}(\bar{\Phi})\,. \tag{2.5.30}$$

(See the slow roll conditions.) It follows that

$$\begin{aligned}\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H^2}{2\pi\partial_t\bar{\Phi}}\right)^2 \approx \left(\frac{3H^3}{2\pi V_{,\bar{\Phi}}}\right)^2 &\approx \frac{(V/M_{\rm Pl}^2)^3}{3(2\pi)^2(V_{,\bar{\Phi}})^2} \\ &= \frac{8}{3}\left(\frac{V^{1/4}}{\sqrt{8\pi}M_{\rm Pl}}\right)^4 \frac{1}{\epsilon_V}\,,\end{aligned} \tag{2.5.31}$$

where, recall, ϵ_V is the slow-roll parameter

$$\epsilon_V \equiv \frac{M_{\rm Pl}^2}{2}\left(\frac{V'}{V}\right)^2. \tag{2.5.32}$$

The large-angle CMB observations constrain $\mathcal{P}_{\mathcal{R}}(k) \sim 2\times 10^{-9}$ on current Hubble scales. It follows that

$$V^{1/4} \sim 6\times 10^{16}\epsilon_V^{1/4}\,{\rm GeV}\,. \tag{2.5.33}$$

The quantity $V^{1/4}$ describes the *energy scale of inflation* and, since $\epsilon_V \ll 1$, the energy scale is at least two orders of magnitude below the Planck scale ($\sim 10^{19}$ GeV). It is, however, plausible, that inflation occurred around the GUT scale, $\sim 10^{16}$ GeV.

F. Spectral index of the primordial power spectrum

We have already noted that slow-roll inflation produces a spectrum of curvature perturbations that is almost scale-invariant. We can quantify the small departures from scale-invariance by forming the *spectral index* $n_s(k)$. Generally, this is a scale-dependent quantity defined by

$$n_s(k) - 1 \equiv \frac{d\ln\mathcal{P}_{\mathcal{R}}(k)}{d\ln k}\,, \tag{2.5.34}$$

where the -1 is conventional and (unfortunately!) means that a scale-free spectrum has $n_s = 1$. For a constant n_s, this definition implies a power-law spectrum

$$\mathcal{P}_{\mathcal{R}}(k) = A_s(k/k_{\rm pivot})^{n_s-1} \tag{2.5.35}$$

for some pivot scale $k_{\rm pivot}$.

We can evaluate n_s by noting that

$$\frac{d}{d\ln k} = \frac{dt}{d\ln k}\frac{d\bar{\Phi}}{dt}\frac{d}{d\bar{\Phi}}\,, \tag{2.5.36}$$

and, since $k = aH$ at Hubble exit,

$$\frac{d\ln k}{dt} = H\left(1+\frac{\partial_t H}{H^2}\right)\,. \tag{2.5.37}$$

The Friedmann equation, and the slow-roll approximation then gives

$$\begin{aligned}\frac{\partial_t H}{H^2} &= -\frac{3}{2}\left(\frac{\bar{\rho}+\bar{P}}{\bar{\rho}}\right)\\ &\approx -\frac{3}{2}\frac{(\partial_t\bar{\Phi})^2}{V}\\ &= -\frac{1}{2}\frac{(3H\partial_t\bar{\Phi})^2}{3H^2V}\\ &\approx -\frac{M_{\rm Pl}^2}{2}\left(\frac{V_{,\bar{\Phi}}}{V}\right)^2 = -\epsilon_V\,,\end{aligned} \tag{2.5.38}$$

so that $d\ln k/dt \approx H(1-\epsilon_V)$. We thus have, to leading-order in the slow-roll parameters,

$$\begin{aligned}\frac{d}{d\ln k} &\approx \frac{1}{H}\frac{d\bar{\Phi}}{dt}\frac{d}{d\bar{\Phi}}\\ &\approx -\frac{V_{,\bar{\Phi}}}{3H^2}\frac{d}{d\bar{\Phi}}\\ &\approx -M_{\rm Pl}^2\frac{V_{,\bar{\Phi}}}{V}\frac{d}{d\bar{\Phi}}\\ &\approx -M_{\rm Pl}\sqrt{2\epsilon_V}\frac{d}{d\bar{\Phi}}\,.\end{aligned} \tag{2.5.39}$$

We can now differentiate Eq. (2.5.31) to find

$$\begin{aligned}n_s - 1 &= -M_{\rm Pl}\sqrt{2\epsilon_V}\frac{d}{d\bar{\Phi}}\left(\ln V - \ln\epsilon_V\right)\\ &= -M_{\rm Pl}\sqrt{2\epsilon_V}\left(\frac{V_{,\bar{\Phi}}}{V} - \frac{\epsilon_{V,\bar{\Phi}}}{\epsilon_V}\right)\,.\end{aligned} \tag{2.5.40}$$

The derivative of ϵ_V is

$$\begin{aligned}\frac{d\ln\epsilon_V}{d\bar{\Phi}} &= 2\left(\frac{V_{,\bar{\Phi}\bar{\Phi}}}{V_{,\bar{\Phi}}} - \frac{V_{,\bar{\Phi}}}{V}\right)\\ &\approx \frac{\sqrt{2}}{M_{\rm Pl}}\left(\frac{\eta_V}{\sqrt{\epsilon_V}} - 2\sqrt{\epsilon_V}\right)\,,\end{aligned} \tag{2.5.41}$$

where η_V is the slow-roll parameter related to the curvature of the potential previously defined in the first part of the course. This gives the final, simple result

$$n_s(k) - 1 = 2\eta_V(\bar{\Phi}) - 6\epsilon_V(\bar{\Phi})\,. \tag{2.5.42}$$

We see that departures from scale-invariance are first order in the slow-roll parameters. It can be shown that $dn_s/d\ln k$ is second-order in slow roll so a power-law primordial power spectrum is a very good approximation for slow-roll inflation.

In this simple class of inflation models, the power spectrum of $\mathcal{R}$ is thus

$$\mathcal{P}_{\mathcal{R}}(k) = \left(\frac{H^2}{2\pi\partial_t\bar{\Phi}}\right)^2_{k=aH} \approx \left(\frac{V^3}{12\pi^2 V_{,\bar{\Phi}}^2}\right)_{k=aH} , \tag{2.5.43}$$

where the second equality uses the slow-roll approximations. The efficiency of this mechanism for producing cosmological curvature perturbations depends on both the height of the potential, which determines the expansion rate and hence size of $\delta\phi$, and its slope, which enters through the conversion of inflaton fluctuations to time delays and so curvature.

Gravitational waves are a direct probe of the Hubble rate during inflation, or, using the slow-roll approximation,

$$\mathcal{P}_h(k) \approx \frac{128}{3}\left(\frac{V^{1/4}}{\sqrt{8\pi}M_{\rm Pl}}\right)^4 , \tag{2.5.44}$$

of the energy scale. Note that

$$r \equiv \frac{\mathcal{P}_h(k)}{\mathcal{P}_{\mathcal{R}}(k)} \approx 16\epsilon_V , \tag{2.5.45}$$

which defines the dimensionless tensor-to-scalar ratio r.

The spectrum of gravitational waves is almost scale-invariant with a spectral index

$$\begin{aligned} n_t \equiv \frac{d\ln\mathcal{P}_h(k)}{d\ln k} \approx \frac{d\ln V}{d\ln k} &= -M_{\rm Pl}\sqrt{2\epsilon_V}\frac{V_{,\bar{\Phi}}}{V} \\ &= -M_{\rm Pl}\sqrt{2\epsilon_V}\frac{\sqrt{2\epsilon_V}}{M_{\rm Pl}} \\ &= -2\epsilon_V . \end{aligned} \tag{2.5.46}$$

Note that this is always negative (the spectrum is said to be *red*) which is a direct consequence of the Hubble parameter falling as inflation proceeds. It follows that $r \approx -8n_t$ in slow-roll inflation which is an example of a slow-roll *consistency relation* between the spectra of curvature perturbations and gravitational waves.

G. Observable predictions and current observational constraints

The inflationary proposal requires a huge extrapolation of the known laws of physics. In the absence of a complete theory, a phenomenological approach has been commonly employed, where an effective potential $V(\phi)$ is postulated. Ultimately, $V(\phi)$ has to be derived from a fundamental theory, and significant progress in implementing inflation in string theory has been made in recent years. However, while it is challenging to understand the origin of inflation from a particle physics point of view, it is also a great opportunity to learn about ultra-high-energy physics from cosmological observations.

The simplest inflationary scenarios consist of a single light scalar field with a canonical kinetic term, $(\nabla\phi)^2/2$, in its action. They predict the following observable characteristics.

1. *Flat geometry*, i.e. the observable universe should have no spatial curvature. As we have seen, flatness has been verified at the 1% level by the location, or, better, separation, of the CMB acoustic peaks combined with some low-redshift distance information.

2. *Gaussianity*, i.e. the primordial perturbations should correspond to Gaussian random variables to a very high precision.

3. *Scale-invariance*, i.e. to a first approximation, there should be equal power at all length-scales in the perturbation spectrum, without being skewed towards high or low wavenumbers. In terms of the parameterisation above, this corresponds to $n_s = 1$ and $n_t = 0$. However, small deviations from scale-invariance are also a typical signature of inflationary models and tell us about the dynamics of inflation.

4. *Adiabaticity*, i.e. after reheating, there are no perturbations in the relative number densities of different species on super-Hubble scales. (so no isocurvature modes). This follows from the assumption that only a single field is important during inflation.

5. *Super-Hubble fluctuations*, i.e. there exist correlations between anisotropies on scales larger than the apparent causal horizon, beyond which two points could not have exchanged information at light-speed during the history of a non-inflationary universe. This corresponds to angular separations on the sky larger than $\sim 2^{\circ}$.

6. *Primordial gravitational waves*, which give rise to temperature and polarization anisotropies. These tensor modes must exist; however, their predicted amplitude can vary by many orders of magnitude depending on the underlying microphysical mechanism implementing inflation.

Please see the handout for a summary of the current observational evidence for inflation.

VI. THE COSMIC MICROWAVE BACKGROUND

Separate notes will be provided for this section.

VII. THE MATTER POWER SPECTRUM

The power spectrum of the late-time distribution of matter is a key cosmological observable. It can be estimated in galaxy surveys by assuming that the fractional fluctuations in the number density of galaxies traces (generally in a *biased* manner) the fractional fluctuations in the matter.

Consider the fractional matter overdensity in the comoving gauge well after recombination, $\Delta_m(\eta, \mathbf{k})$. The matter power spectrum $\mathcal{P}_{\Delta_m}(\eta; k)$ is then defined by

$$\langle \Delta_m(\eta, \mathbf{k}) \Delta_m^*(\eta, \mathbf{k}') \rangle \equiv \frac{2\pi^2}{k^3} \mathcal{P}_{\Delta_m}(\eta; k) \delta(\mathbf{k} - \mathbf{k}') \,. \tag{2.5.47}$$

The power spectrum $\mathcal{P}_{\Delta_m}(\eta; k)$ is dimensionless, but frequently a dimensional spectrum $P_{\Delta_m}(\eta; k)$ is used, where

$$P_{\Delta_m}(\eta; k) \equiv \frac{2\pi^2}{k^3} \mathcal{P}_{\Delta_m}(\eta; k) \,. \tag{2.5.48}$$

One may also meet real-space measures of matter clustering, such a σ_R. This is defined to be the (real-space) variance of Δ_m averaged in spheres of radius R. This is equivalent to the variance of Δ_m convolved with $3\Theta(R - |\mathbf{x}|)/(4\pi R^3)$, i.e. a normalised spherical top-hat of radius R. The Fourier transform of such a top-hat is $W(kR)$, where

$$W(x) \equiv \frac{3}{x^3} (\sin x - x \cos x) \,. \tag{2.5.49}$$

From the convolution theorem, the variance of the convolved field is

$$\sigma_R^2 = \int \frac{d^3\mathbf{k}}{(2\pi)^3} W^2(kR) P_{\Delta_m}(k) \,. \tag{2.5.50}$$

Historically, a scale $R = 8h^{-1}\,\mathrm{Mpc}$ is chosen, where the Hubble constant $H_0 = 100h\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, since $\sigma_8 \sim 1$. The smoothing scale at which the variance of a low-pass-filtered field exceeds unity — here $\sim 8h^{-1}\,\mathrm{Mpc}$ — marks the scale at which perturbation theory breaks down and non-linear effects become important. In the best-fit model to the WMAP CMB data, $\sigma_8 \approx 0.80$.

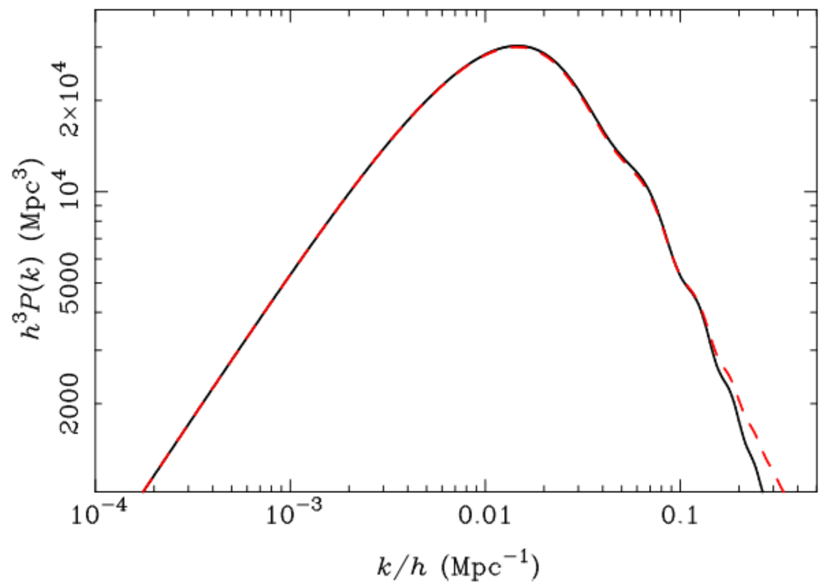

FIG. 5 The matter power spectrum $P_{\Delta_m}(k)$ at $z = 0$ in linear theory (solid) and with non-linear corrections (dashed). Credit: A. Challinor

The theoretical matter power spectrum at $z = 0$ is plotted in Fig. 5 based on linear perturbation theory. Also plotted is a fit to the results of N-body simulations that takes account of the non-linear growth of the density field on small scales. The main features of this plot are as follows.

- On large scales, $P_{\Delta_m}(k)$ grows as k.
- The power spectrum turns over around $k \sim 0.01\,\mathrm{Mpc}^{-1}$ corresponding to the horizon size at mattter-radiation equality.
- Beyond the peak, the power falls as $k^{-3}\ln^2(k/k_{\rm eq})$, where $k_{\rm eq}$ is the wavenumber of a mode that enters the horizon at the matter-radiation transition.
- There are small amplitude baryon acoustic oscillations in the spectrum (see also Fig. 3).
- Linear theory applies on scales $k^{-1} > 10\,\mathrm{Mpc}$ at $z = 0$.

The *shape* of the matter power spectrum as inferred from galaxy clustering agrees well with the theoretical prediction in Fig. 5. Inferring the amplitude is more problematic due to the issue of galaxy bias noted above.

To understand the shape of the matter power spectrum, we note that on scales where linear perturbation theory holds we can write (for adiabatic initial conditions)

$$\Delta_m(\eta, \mathbf{k}) = T(\eta, k)\mathcal{R}(0, \mathbf{k})\,, \tag{2.5.51}$$

where $T(\eta, k)$ is the transfer function which relates the primordial curvature perturbation to the comoving matter perturbation. The primordial curvature power spectrum is almost scale-free ($\mathcal{P}_{\Delta_m}(0; k) \approx$ const.; see Sec. V) so it contributes a factor of k^{-3} to $P_{\Delta_m}(k)$. This primordial shape is modified by the transfer function whose scale dependence results from the physics of structure formation.

First consider modes with $k < k_{\rm eq}$. These were outside the horizon throughout radiation domination. For these, it turns out to be simplest to consider the Newtonian-gauge potential Φ: during matter domination $\Phi(\eta, \mathbf{k}) = -3\mathcal{R}(0, \mathbf{k})/5$ for $k < k_{\rm eq}$. The potential is suppressed uniformly on all scales when Λ comes to dominate, so that $\Phi(\eta, \mathbf{k})/\mathcal{R}(0, \mathbf{k})$ remains independent of k. We can relate Δ_m to Φ via the Poisson equation which, in Fourier space, gives $\Delta_m \sim k^2\Phi$. It follows that

$$T(\eta, k) \propto k^2 \qquad k < k_{\rm eq}\,, \tag{2.5.52}$$

and so $P_{\Delta_m}(k) \propto k^4/k^3 \propto k$ on large scales.

For $k > k_{\rm eq}$, the mode entered the horizon during radiation domination. Generally, on sub-Hubble scales, the density contrast in the comoving and Newtonian gauges are the same so we can work with δ_m rather than Δ_m. Since CDM dominates the total matter, we shall simplify further by considering the evolution of δ_c. The Newtonian-gauge δ_c is constant in time until horizon entry and, moreover, $\delta_c(0, \mathbf{k}) \propto \Phi(0, \mathbf{k}) \propto \mathcal{R}(0, \mathbf{k})$ so all modes have approximately the same variance at horizon entry. The Meszaros effect operates inside the horizon during radiation domination and δ_c then grows logarithmically with proper time t. After matter-radiation equality, δ_c grows as a. Shorter wavelength modes enter the horizon earlier and have had more logarithmic growth by the end of the radiation era than longer modes. The ratio

$$\frac{\delta_c(t_{\rm eq}, \mathbf{k})}{\delta_c(0, \mathbf{k})} \sim 1 + \ln(t_{\rm eq}/t_k) \qquad k > k_{\rm eq} \tag{2.5.53}$$

due to the Meszaros effect, where $t_{\rm eq}$ is the time of matter-radiation equality and t_k is the time of horizon entry. Since $a(t_k)H(t_k) = k$, and $a \propto t^{1/2}$ in radiation domination, $t_k \propto 1/k^2$. The Meszaros enhancement is thus $\sim 1+2\ln(k/k_{\rm eq})$ for $k > k_{\rm eq}$. After the end of the radiation era, δ_c grows uniformly on all scales (expect for the small scale-dependent effect of the baryons which gives rise to the baryon acoustic oscillations in $P_{\Delta_m}(k)$) and so at late times the k-dependence of the transfer function is

$$T(\eta, k) \propto \ln(k/k_{\rm eq}) \qquad k \gg k_{\rm eq}\,. \tag{2.5.54}$$

Finally, this gives

$$P_{\Delta_m}(k) \propto k^{-3}\ln^2(k/k_{\rm eq}) \qquad k \gg k_{\rm eq}\,. \tag{2.5.55}$$