2 Electrostatics

2.1 Electric charge

Many very simple experiments show the existence of electric charges and forces. For example:

- after running a comb through your hair, it will attract bits of paper;
- after rubbing an inflated baloon with wool, it will adhere to the walls for a long time.

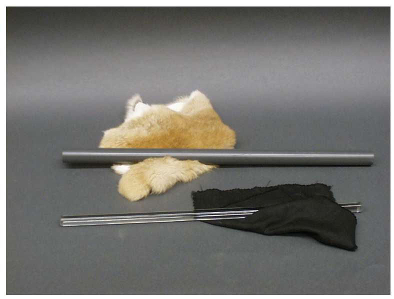


Figure 1: Rub the plastic rod with fur to negatively charge the rod. Rub the glass rod with silk to positively charge the rod.

To be more precise, consider the following two situations (Fig. 2.1). First, a plastic rod is rubbed with fur. Second, a glass road is rubbed with silk. In the first case electrons (the elementary negative charge) are transfered from the fur to the rod, so that the plastic rod becomes **negatively charged**. In the second case, electrons are transfered from the glass rod to the silk, so that the rod becomes **positively charged**. It is important to note that in the process of charging the objects, the electrons get redistributed and are not created or destroyed. This is a general property: **the total charge of an isolated system is conserved**. The simple system of plastic and glass rods can be used to show the existence of electric forces and to demonstrate that there are actually two different kinds of charges (positive and negative, as already mentioned). Indeed, by bringing two rods together it is easy to verify that:

- two charged glass rods repel each other,
- two charged plastic rods repel each other,
- a plastic rod and a glass rod attract each other.

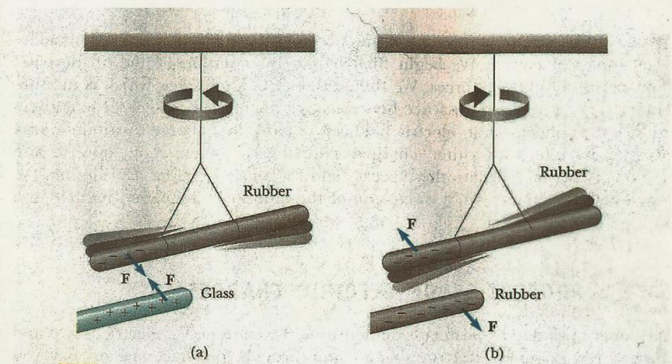


Figure 1 (a) A negatively charged rubber rod suspended by a thread is attracted to a positively charged glass rod. (b) A negatively charged rubber rod is repelled by another negatively charged rubber rod.

This shows the **existence of two different charges**, and the existence of **electric forces** between charged objects. Furthermore, we observe that: **like charges repel while unlike charges attract each other**.

The SI unit for the electric charge is the **coulomb (C)**. The smallest amount of charge e known in nature is the charge of an electron (-e) or of a proton ($+e$), and it is equal to

$$e = 1.602 \cdot 10^{-19} C.$$

The electric charge is **quantized**, and an object can only carry a charge q multiple of the elementary charge e: $q = Ne$ (with N a positive or negative integer).

2.2 Coulomb's law

Coulomb's law states that the force $\underline{F}_{21}$ between two charges q_1 and q_2 at a distance $\underline{r}_{21}$ is:

$$\boxed{\underline{F}_{21} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{21}^2} \hat{r}_{21} \qquad (2.1)}$$

where $\hat{r}_{21}$ is the unit vector between the two charges q_1 and q_2. The constant ϵ_0 is called the **permittivity of free space** and is equal to

$$\epsilon_0 = 8.85 \times 10^{-12} C^2 N^{-1} m^{-2}. \qquad (2.2)$$

2.3 The electric field

In electromagnetism it is very convenient to introduce the concept of **electric field**. We recall that by a 'field' we mean a quantity whose value depends on position in space.

Consider once again two charges, q_1 and q_2, at a distance $\underline{r}_{21}$. We have seen that the interaction between the two charges is described by the Coulomb law, which predicts a force equal to

$$\underline{F}_{21} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{21}^2}\hat{r}_{21} \ .$$

In the approach of fields, we say that one of the charges (say q_1) creates an **electric field** $\underline{E}$ in space:

$$\underline{E} = \frac{1}{4\pi\epsilon_0}\frac{q_1}{r^2}\hat{r} \ . \tag{2.3}$$

When another charge (in this case q_2) is introduced, an electric force acts on it. This force is given by:

$$\underline{F}_{21} = q_2\underline{E} \ , \tag{2.4}$$

and we recover Coulomb's law. In other words, the electric field can be defined as **the electric force acting on a charge at a point in space divided by the magnitude of the charge.**

The electric field has units of newtons per coulomb (N/C).

Consider now the problem of determining the electric field generated not by just one charge, but by a group of charges. The electric field can be easily calculated by applying the superposition principle: **the total electric field due to a group of charges equals the vector sum of the electric fields of all the charges**. Therefore, the total electric field $\underline{E}(P)$ at the point P due to the charges $q_1, q_2 ...., q_n$ is:

$$\underline{E} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r}_i \tag{2.5}$$

where $\underline{r}_i$ is the distance from the position of the charge q_i to the point P.

Consider now the case of a continuous charge distribution. Suppose that within a volume V there is a **charge density** $\rho = \rho(x, y, z)$. This means that at the point (x, y, z) there is a charge of ρ per unit volume.

The charge in an infinitesimal volume dV is then $dq = \rho dV$ and the infinitesimal electric field produced by this charge at the point $\underline{R}$ is then

$$\underline{dE}(\underline{R}) = \frac{1}{4\pi\epsilon_0} \frac{\underline{R} - \underline{r}}{|\underline{R} - \underline{r}|^3} \rho(\underline{r}) dV \tag{2.6}$$

where $\underline{r} = (x, y, z)$ is the position of the infinitesimal charge $dq = \rho dV$. The total electric field is then obtained by direct integration over the volume V

$$\boxed{\underline{E}(\underline{R}) = \int_V \frac{1}{4\pi\epsilon_0} \frac{\underline{R} - \underline{r}}{|\underline{R} - \underline{r}|^3} \; \rho(\underline{r}) dV \ . \qquad (2.7)}$$

The treatment above applies also to the case of a charge distributed over a surface or over a line. In the first case $dq = \sigma \mathrm{d}a$, with σ the **surface charge density**, and $\mathrm{d}a$ the infinitesimal area element. In the second case $\mathrm{d}q = \lambda dl$, with λ the **linear charge density** and dl an infinitesimal length element.

Exercise: A charge Q is uniformly distributed over a disk of radius R and axis Oz. Determine the electric field at a point P on the z axis.

Exercise: Determine the electric field created by a segment of length L, carrying a linear charge density λ, at a point P located on the medium plane of the wire.

Electric fields can be represented pictorially by **electric field lines**. These lines are parallel to the electric field vector at any point in space. The basic properties of these lines are:

- The lines must begin on a positive charge and terminate on a negative charge. If the total charge of the system is non-zero, some lines will begin or end infinitely far away.
- $\underline{E}$ is tangent to the electric field line at each point. The direction of the line is the same of that of $\underline{E}$;
- the number of lines per unit area through a surface perpendicular to the field lines is proportional to the magnitude of the electric field in that region of space.

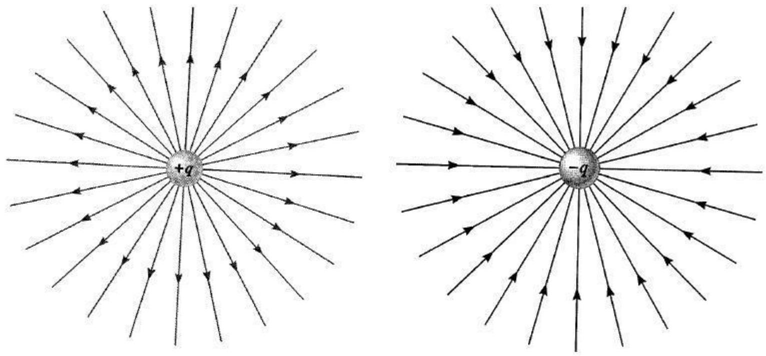


Figure 2: Field lines for a single charge (positive and negative).

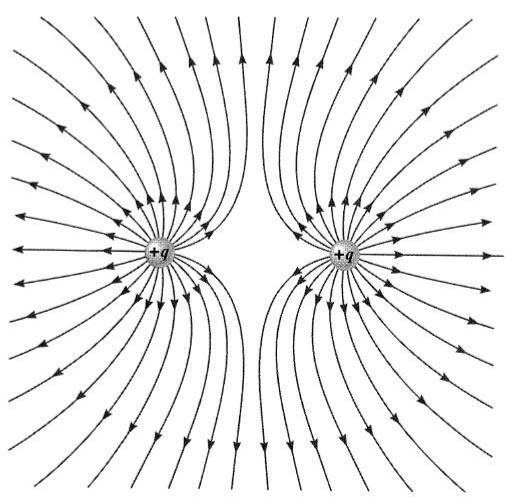


Figure 3: Field lines due to two equal positive charges.

2.4 Gauss' law

Gauss' law relates the electric field on any **closed** surface to the net amount of charge enclosed within the surface. We will see that Gauss' law is very useful to determine the electric field produced by a charge distribution with a simple geometry.

In order to derive Gauss' law we first introduce two concepts: the flux of a vector field, and the solid angle.

1. Flux of a vector field

   The idea of flux of a vector field is easily explained for a fluid. In this case the vector field is the velocity $\underline{v}$. Consider a small area δa perpendicular to the direction of flow of the fluid (see Fig. 4, left). The fluid flux is the rate of flow of the fluid through the area, which is

$$v\delta a \ . \tag{2.8}$$

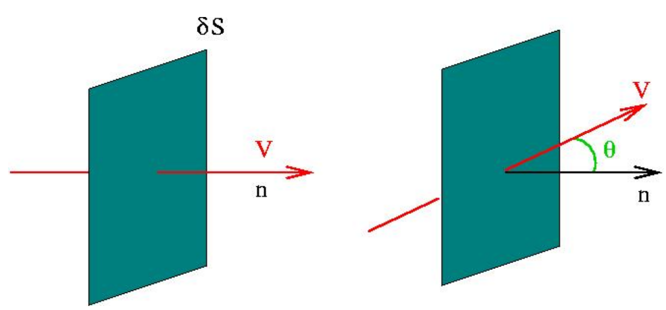


Figure 4: Definition of the flux of a vector field

   If the small area is not perpendicular to v, we have to consider the projection of δa onto the plane perpendicular to the vector $\underline{v}$ (see Fig. 4), right. Such a projection is equal to $\delta a \cos\theta$. Using vectorial notation, the flux is equal to

$$\underline{v} \cdot \hat{n}\delta a \ , \tag{2.9}$$

   where $\hat{n}$ is the unit vector normal to the surface δa. We note that the sense of the unit vector has to be specified, as $\hat{n}$ and $-\hat{n}$ lead to fluxes of same magnitude but opposite sign.

   For a curved surface, it is necessary to split the surface in lot of small flat surfaces and then sum over these surfaces, i.e. we have to consider the sum

$$\sum \underline{v} \cdot \underline{n}_i \delta a_i \ , \tag{2.10}$$

which in the limit $\delta a \to 0$ becomes

$$\Phi_v = \int_a \underline{v} \cdot \hat{n} \; da \; . \tag{2.11}$$

Equation (2.11) generalizes to any vector field, and in particular applies to the electric field $\underline{E}$ whose flux through a surface S is

$$\boxed{\Phi_E = \int_a \underline{E} \cdot \hat{n} \; da \; . \qquad (2.12)}$$

2. Solid angle

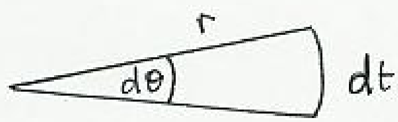
In two dimensions, if we have an arc of a circle of radius r subtended by an angle $\mathrm{d}\theta$, the length of that arc is

$$\mathrm{d}t = r\mathrm{d}\theta.$$

We can use this relation to define the angle $\mathrm{d}\theta$ as

$$\mathrm{d}\theta = \mathrm{d}t/r.$$

Note that since $\mathrm{d}t$ and r both have dimensions of length, the angle $\mathrm{d}\theta$ is a dimensionless quantity.

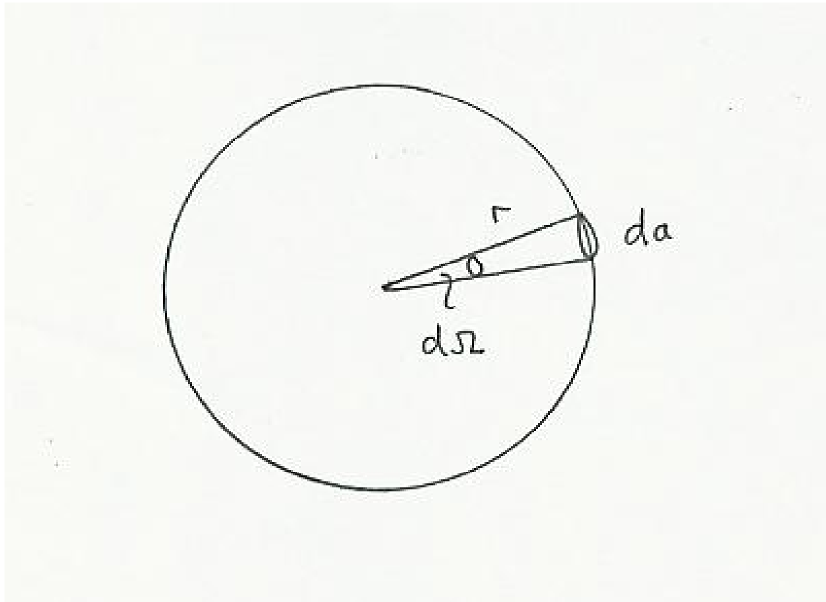


In analogous way we can define a dimensionless solid angle $\mathrm{d}\Omega$ in terms of an element of area, $\mathrm{d}a$, subtended on the surface of a sphere

$$\boxed{\mathrm{d}\Omega = \mathrm{d}a/r^2 \qquad (2.13)}$$

If we integrate over the surface of the whole sphere we see that

$$\Omega = \int_{\text{sphere}} \mathrm{d}\Omega = \frac{1}{r^2}\int_{\text{sphere}} \mathrm{d}a = \frac{1}{r^2}\, 4\pi r^2 = 4\pi \tag{2.14}$$



3. Derivation of Gauss' law

Now consider a point charge q surrounded by a closed surface, S, of arbitrary shape. From Coulomb's law we know that the electric field vector $\underline{E}$ is directed radially outwards from the charge. Consider an infinitesimal area da on this surface. The unit vector, $\hat{\underline{n}}$ normal to this surface will in general not be in the radial direction. Define a vector area d$\underline{a}$ by

$$\mathrm{d}\underline{a} = \hat{\underline{n}}\ \mathrm{d}a \tag{2.15}$$

ie a vector of magnitude da in the direction of the normal to the surface.

The electric field due to the charge q is the usual

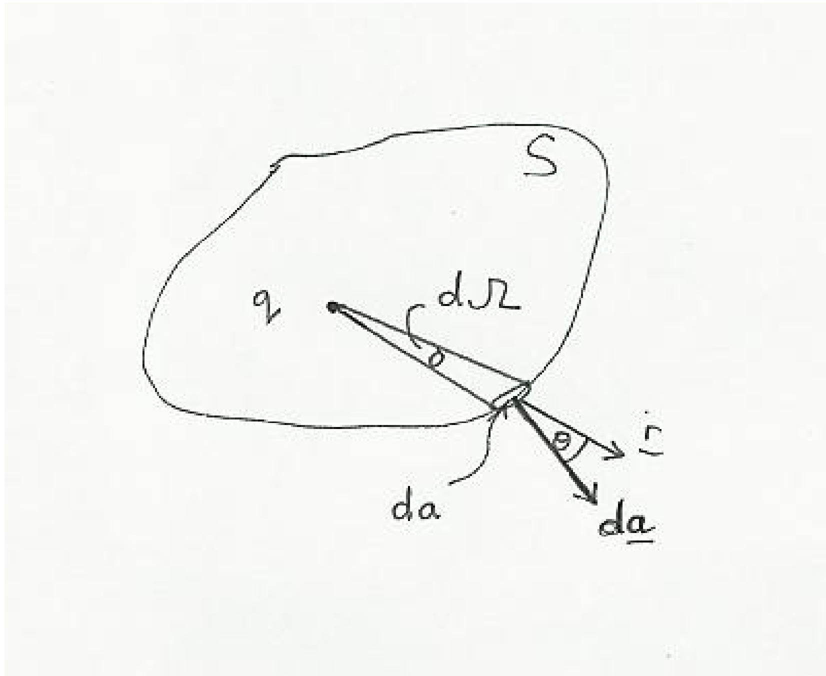
$$\underline{E} = \frac{q}{4\pi\epsilon_0 r^2}\ \hat{\underline{r}} \tag{2.16}$$

For an arbitrary shape of surface, $\underline{E}$ and d$\underline{a}$ will not have the same direction.

Now consider the total flux of the electric field through the surface S given by the integral

$$\int_S \underline{E}\cdot \mathrm{d}\underline{a} = \frac{q}{4\pi\epsilon_0}\int_S \hat{\underline{r}}\cdot\hat{\underline{n}}\ \frac{\mathrm{d}a}{r^2} \tag{2.17}$$

$$= \frac{q}{4\pi\epsilon_0}\int_S \frac{\cos\theta}{r^2}\ \mathrm{d}a \tag{2.18}$$



$\mathrm{d}a\cos\theta = \mathrm{d}a'$ is the area $\mathrm{d}a$ projected perpendicular to $\hat{\underline{r}}$.

Now the solid angle element $\mathrm{d}\Omega$ is defined by

$$\mathrm{d}\Omega = \frac{\mathrm{d}a'}{r^2} \tag{2.19}$$

Hence

$$\int_S \underline{E} \cdot \mathrm{d}\underline{a} = \frac{q}{4\pi\epsilon_0} \int_S \mathrm{d}\Omega = \frac{q}{4\pi\epsilon_0}\, 4\pi = \frac{q}{\epsilon_0} \tag{2.20}$$

In the case of many charges contained within the surface, $q_1, q_2, \ldots q_n$, each gives rise to an electric field $\underline{E}_1, \underline{E}_2, \ldots \underline{E}_n$. By applying the principle of superposition, we conclude that the total electric field, $\underline{E}$, at any point is the vector sum

$$\underline{E} = \sum_1^n \underline{E}_i \ . \tag{2.21}$$

Then evaluating the same integral as above,

$$\int_S \underline{E} \cdot \mathrm{d}\underline{a} = \sum_i \int_S \underline{E}_i \cdot \mathrm{d}\underline{a} \tag{2.22}$$

$$= \sum_i \frac{q_i}{4\pi\epsilon_0} \int_S \hat{\underline{r}}_i \cdot \hat{\underline{n}} \, \frac{\mathrm{d}a}{r_i^2} \tag{2.23}$$

$$= \sum_i \frac{q_i}{4\pi\epsilon_0} \int_S \mathrm{d}\Omega_i \tag{2.24}$$

$$= \sum_i \frac{q_i}{\epsilon_0} \tag{2.25}$$

$$\boxed{\int_S \underline{E} \cdot \mathrm{d}\underline{a} = \frac{Q_{\mathrm{internal}}}{\epsilon_0},} \tag{2.26}$$

where Q_{internal} is the sum of all charges within the surface S. Equation 2.26 is Gauss' law of electrostatics.

Note that the flux through the surface a does not depend on the shape or size of the surface, only on the amount of charge it contains.

We have just derived Gauss' law for a collection of discrete charges within a surface S. We wish to extend the law to a 'continuous' charge distribution. Suppose that within a volume V there is a charge density $\rho = \rho(x, y, z)$. This means that at the point (x, y, z) there is a charge of ρ per unit volume.

The charge in an infinitesimal volume $\mathrm{d}V$ is then $\rho\mathrm{d}V$ and the total charge in the volume V is an integral

$$Q_{\mathrm{internal}} = \int_V \rho \mathrm{d}V \tag{2.27}$$

Gauss' law for a continuous charge distribution is then

$$\boxed{\int_S \underline{E} \cdot \mathrm{d}\underline{a} = \int_V \frac{\rho}{\epsilon_0} \, \mathrm{d}V} \tag{2.28}$$

2.5 Electrostatics in simple geometries

Now let us apply Gauss' law to some simple examples of charges or charge distributions with 1) spherical 2) cylindrical or 3) planar geometry.

1. <u>Spherical</u>

(a) <u>A uniformly charged solid sphere</u>

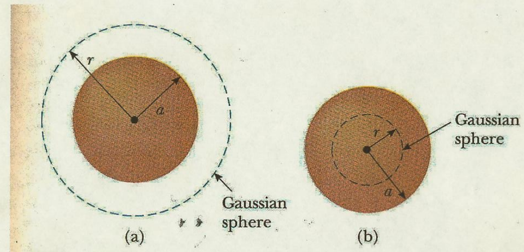


Figure 15 A uniformly charged insulating sphere of radius a and total charge Q. (a) The magnitude of the electric field at a point exterior to the sphere is $k_e Q/r^2$. (b) The magnitude of the electric field inside the insulating sphere is due only to the charge *within* the gaussian sphere defined by the dashed circle and is $k_e Q r/a^3$.

Consider a sphere (radius a) with a uniform charge density ρ. Then the total charge on the sphere is

$$Q = \frac{4}{3}\,\pi a^3\,\rho \tag{2.29}$$

First, consider a spherical surface S that is concentric with and encloses the charged sphere, with radius $r \geq a$. From the symmetry of the system we can see that the electric field is directed radially outwards and is the same everywhere on the surface S. Let the magnitude of the electric field be E.
The LHS of equation 2.26 (Gauss' law) is then

$$\int_S \underline{E} \cdot \mathrm{d}\underline{a} = \int_S E\ \mathrm{d}a \tag{2.30}$$

$$= E \int_S \mathrm{d}a \tag{2.31}$$

$$= E\ 4\pi r^2 \tag{2.32}$$

The RHS of equation 2.26 is just $\frac{Q}{\epsilon_0}$, so that

$$4\pi r^2 E = \frac{Q}{\epsilon_0} \tag{2.33}$$

therefore

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} \qquad (r \geq a) \qquad (2.34)$$

which is the same as for a point charge Q.

Second, choose a surface S, inside the sphere, with radius $r < a$. Now the RHS of 2.26 is

$$\int_V \frac{\rho}{\epsilon_0} \mathrm{d}V = \frac{\rho}{\epsilon_0} \, \frac{4}{3} \, \pi r^3 \qquad (2.35)$$

while the LHS is the same as in 2.32 so that

$$4\pi r^2 \, E = \frac{4}{3}\pi r^3 \frac{\rho}{\epsilon_0} \qquad (2.36)$$

therefore

$$E = \frac{\rho r}{3\epsilon_0}, \qquad r < a \qquad (2.37)$$

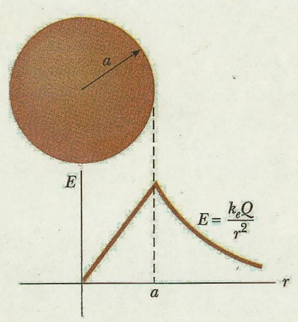


Figure 16 A plot of E versus r for a uniformly charged insulating sphere. The electric field inside the sphere ($r < a$) varies linearly with r. The field outside the sphere ($r > a$) is the same as that of a point charge Q located at $r = 0$.

so the electric field varies linearly with distance from the centre of the charged sphere.

(b) A hollow sphere

Consider a thin spherical shell of radius a and negligible thickness, with a constant charge per unit area, σ.

The field outside the shell is the same as for a point charge or a solid sphere:

$$E = \frac{1}{4\pi\epsilon_0} \; \frac{Q}{r^2} \qquad (r \geq a) \tag{2.38}$$

where $Q = 4\pi a^2 \sigma$ is the total charge on the shell.

Inside the shell the field is zero, as the RHS of 2.26 is zero.

2. Cylindrical geometry

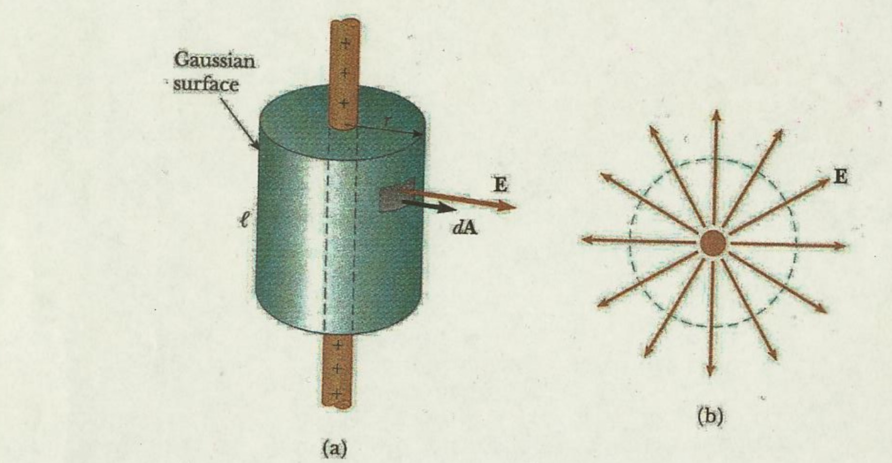


Figure 18 (a) An infinite line of charge surrounded by a cylindrical gaussian surface concentric with the line. (b) An end view shows that the electric field at the cylindrical surface is constant in magnitude and perpendicular to the surface.

Consider an infinitely long line of charge (eq a wire with negligible diameter) with a constant charge per unit length λ.

Take a cylindrical surface S, with radius r, length l about this line.

By symmetry, $\underline{E}$ must be perpendicular to the line of charge, directed radially outwards. Also E will be the same at all points on the curved part of S. Thus, for the top and bottom surfaces of S, the LHS of 2.26 is zero since $\underline{E} \cdot \underline{\hat{n}} = 0$ there. So the LHS of 2.26 is

$$\int_S \underline{E} \cdot \mathrm{d}\underline{a} = E \;\; 2\pi r l \tag{2.39}$$

The RHS of 2.26 is

$$\frac{Q}{\epsilon_0} = \frac{\lambda l}{\epsilon_0} \tag{2.40}$$

so

$$2\pi r l \;\; E = \frac{\lambda l}{\epsilon_0} \tag{2.41}$$

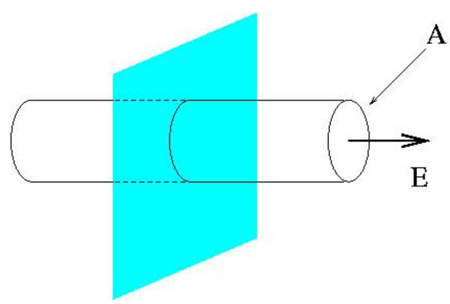
and

$$\boxed{E = \frac{1}{2\pi r l} \;\; \frac{\lambda l}{\epsilon_0} = \frac{\lambda}{2\pi\epsilon_0 r}. \qquad (2.42)}$$

Thus E falls of as the inverse of the distance from the wire.

3. Planar geometry

   Consider an infinite plane of negligible thickness, with constant charge per unit area, σ.



Take a cylindrical surface S (see Figure) with top and bottom faces of area A.

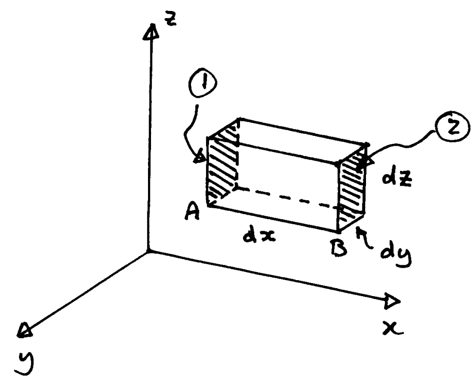
By symmetry, the flux is zero through the curved surfaces of S, so the LHS of 2.26 is 2 E A. The RHS is $\frac{Q}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$. Therefore

$$\boxed{E = \frac{\sigma}{2\epsilon_0}. \qquad (2.43)}$$

So the field strength is constant and does not depend on the distance from the (infinite) sheet.

2.6 Differential form of Gauss' law

Apply Gauss' law to an infinitesimal volume and shrink to obtain a law that applies at a point Consider the elementary volume shown. Suppose there is a charge density ρ (charge per unit volume) in this region. The total charge in the elementary volume is then $\rho \mathrm{d}x\mathrm{d}y\mathrm{d}z$.



Suppose there is an electric field $\underline{E}$ with components E_x, E_y, E_z at A. The field at B is then

$$E_x + \mathrm{d}E_x = E_x + \left(\frac{\partial E_x}{\partial x}\right)\mathrm{d}x \tag{2.44}$$

At all points on the surface (2), the x-component of E will be greater than that at corresponding points on surface (1) (points with the same y, z) by $\left(\frac{\partial E_x}{\partial x}\right)\mathrm{d}x$.

The net flux of $\underline{E}$ through the surfaces (1) and (2) is

$$\left(E_x + \frac{\partial E_x}{\partial x}\mathrm{d}x\right)\mathrm{d}y\mathrm{d}z - E_x\mathrm{d}y\mathrm{d}z = \left(\frac{\partial E_x}{\partial x}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{2.45}$$

as $\delta\underline{a}$ has been taken outward normal to the surface.

We can make the same argument for the other two pairs of faces on the volume. We then get for the total net outward flux from the volume

$$\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z \tag{2.46}$$

From Gauss' law this must equal the charge inside the volume divided by ϵ_0

$$\frac{\rho\mathrm{d}x\mathrm{d}y\mathrm{d}z}{\epsilon_0}$$

therefore

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0} \tag{2.47}$$

The LHS is the divergence of $\underline{E}$, therefore

$$\text{div}\underline{E} = \underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0} \tag{2.48}$$

This is Gauss' law of electrostatics in differential form.

Since the net outward flux from the volume $\mathrm{d}x\mathrm{d}y\mathrm{d}z$ was div$\underline{E}$ $\mathrm{d}x\mathrm{d}y\mathrm{d}z$, div$\underline{E}$ must be the flux per unit volume at a point. If there is no source of charge ($\rho = 0$), div$\underline{E} = 0$ and there is not net flux. Hence, div$\underline{E}$ is only non-zero if there is a source or sink of field lines at that point.

2.7 Electric potential

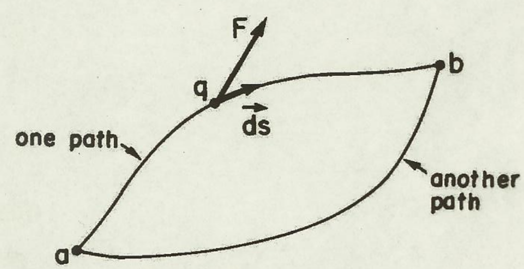


Fig. 20 The work done in carrying a charge from *a* to *b* is the negative of the integral of **F · ds** along the path taken.

Consider the work done in carrying a charge, q_0, from point a to point b in an electric field (see Figure). It is the negative of the integral of $\underline{F} \cdot \mathrm{d}\underline{s}$ along the path taken

$$W = -\int_a^b \underline{F} \cdot \mathrm{d}\underline{s} \tag{2.49}$$

(negative because we do work when we move against the force, eg if $\underline{\mathrm{F}}$ and $\mathrm{d}\underline{s}$ are in opposite directions, we do work and the potential energy increases.)

Now the force and the electric field are related by

$$\underline{F} = q_0 \, \underline{E}, \tag{2.50}$$

so that

$$\boxed{W = -q_0 \int_a^b \underline{E} \cdot \mathrm{d}s. \qquad (2.51)}$$

This is the work done against the electric force to move the charge q_0. As energy is conserved, this must equal the change in potential energy U of the charge-plus-field system.

Now consider the work done on a unit charge (set q_0 to unity):

$$W_{\text{unit}} = -\int_a^b \underline{E} \cdot \mathrm{d}\underline{s}. \tag{2.52}$$

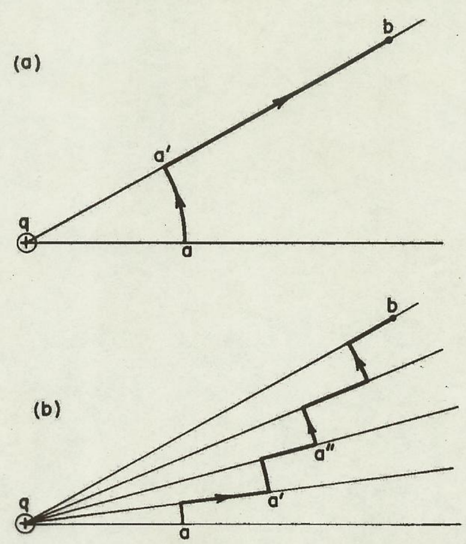


Fig. 21 In carrying a test charge from *a* to *b* the same work is done along either path.

For the case of the field due to a single charge q, we know that

$$\underline{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\,\hat{\underline{r}} \tag{2.53}$$

so that

$$W_{\text{unit}} = -\int_a^b \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\,\hat{\underline{r}}\cdot \mathrm{d}\underline{s} = -\frac{q}{4\pi\epsilon_0}\int_a^b \frac{\mathrm{d}r}{r^2} \tag{2.54}$$

(since $\hat{\underline{r}}\cdot \mathrm{d}\underline{s} = \mathrm{d}r$).
Then

$$\boxed{W_{\text{unit}} = -\frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right), \qquad (2.55)}$$

which depends only on the endpoints of the movement. We deduce from the principle of superposition that this is true in all cases for an electric field. The line integral does not depend on the path taken from a to b.

The RHS of 2.55 is the difference between two numbers. We can write it as

$$W_{\text{unit}} = -\int_a^b \underline{E}\cdot \mathrm{d}\underline{s} = V(b) - V(a) \tag{2.56}$$

where $V(x,y,z)$ is the electric potential at any given point (x,y,z). Note that equation 2.56 only defines the difference between $V(a)$ and $V(b)$, not the absolute value of either.

For convenience, we therefore choose a reference point P_0 (often taken as infinity) where we define $V = 0$. We can then write the potential at a point P as

$$\boxed{V(P) = -\int_{\mathrm{P}_0=\infty}^{\mathrm{P}} \underline{E}\cdot \mathrm{d}\underline{s}. \qquad (2.57)}$$

This is therefore the work done in bringing a unit charge from infinity to the point P through an electric field E.

The units of electric potential are therefore joules per coulomb or volts.

Note that we can choose the path we take from ∞ to P_0, the result will be the same. In particular if part of the path is perpendicular to $\underline{E}$ then the integral over that part is zero, while if the path is parallel to $\underline{E}$, the contribution is just $\int E\mathrm{d}s$.

2.8 Electric field as gradient of the potential

From mechanics we know that the relation between a force $\underline{F}$ and the potential energy associated with it, W, is

$$\underline{F} = -\underline{\nabla} W \tag{2.58}$$

In the case of an electric field $\underline{E}$, the force on a charge is $\underline{F} = q\underline{E}$ and the electric potential is $V = W/q$, the work or energy per unit charge. Therefore

$$q\underline{E} = -\underline{\nabla}(qV) \tag{2.59}$$

therefore

$$\boxed{\underline{E} = -\underline{\nabla} V \qquad (2.60)}$$

In the previous Sections we introduced the concept of *field lines*, useful to display the electric field. For the electric potential, we introduce here the concept of **equipotential surfaces**, i.e. surfaces characterized by the same potential. We notice that field lines always cross equipotential surfaces **orthogonally**, in the direction in which the potential decreases most rapidly (since $\underline{E} = -\underline{\nabla} V$).

Remark: the explicit expression for Eq. 2.60 in the various systems of coordinates is:

in cartesian coordinates x, y, z:

$$\underline{E} = -\left(\frac{\partial V}{\partial x}\hat{x} + \frac{\partial V}{\partial y}\hat{y} + \frac{\partial V}{\partial z}\hat{z}\right)$$

in cylindrical coordinates ρ, ϕ, z:

$$\underline{E} = -\left(\frac{\partial V}{\partial \rho}\hat{u}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\hat{u}_\phi + \frac{\partial V}{\partial z}\hat{u}_z\right)$$

in spherical coordinates r, θ, ϕ:

$$\underline{E} = -\left(\frac{\partial V}{\partial r}\hat{u}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{u}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{u}_\phi\right)$$

2.9 Electric potential for a point charge

For a point charge 2.55 to 2.57 give (as $r_a \to \infty$)

$$\boxed{V(x, y, z) = \frac{q}{4\pi\epsilon_0}\,\frac{1}{r} \qquad (2.61)}$$

where $r = \sqrt{x^2 + y^2 + z^2}$.

The equipotential surfaces (V = constant) about the point charge are spheres.

2.10 Electric potential for a discrete charge distribution

For a point charge we know that

$$V = \frac{q}{4\pi\epsilon_0}\,\frac{1}{r} \tag{2.62}$$

and

$$\underline{E} = -\underline{\nabla} V = \frac{q}{4\pi\epsilon_0}\,\frac{1}{r^2}\,\underline{\hat{r}} \tag{2.63}$$

where V is the potential at a distance r from the charge q.

For the general case of the potential due to a collection of point charges, we consider the potential at some point (x_i, y_i, z_i) due to a set of charges q_j at (x_j, y_j, z_j). We use the principle of superposition:

$$V(x_i, y_i, z_i) = \frac{1}{4\pi\epsilon_0} \sum_j \frac{q_j}{r_{ij}} \tag{2.64}$$

where $r_{ij} = |\underline{r}_{ij}| = |\underline{r}_i - \underline{r}_j|$.

2.11 Electric dipole

A dipole is a system with two charges of equal magnitude and opposite sign separated by a distance $2d$.

The potential at point P is

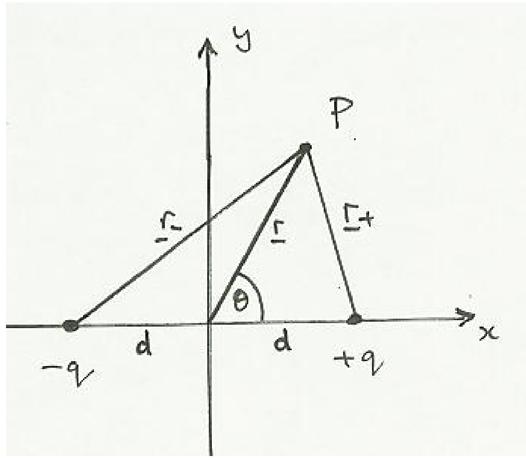
$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \tag{2.65}$$

where, from the cosine rule,

$$r_+^2 = r^2 + d^2 - 2dr\cos\theta \tag{2.66}$$

$$r_-^2 = r^2 + d^2 - 2dr\cos(\pi - \theta) \tag{2.67}$$

$$= r^2 + d^2 + 2dr\cos\theta \tag{2.68}$$



therefore

$$\frac{1}{r_\pm} = (r_\pm^2)^{-\frac{1}{2}} = \left[r^2 \left(1 + \frac{d^2}{r^2} \mp \frac{2d\cos\theta}{r} \right) \right]^{-\frac{1}{2}} \tag{2.69}$$

$$\boxed{\frac{1}{r_\pm} = \frac{1}{r}\left(1 + \frac{d^2}{r^2} \mp \frac{2d\cos\theta}{r} \right)^{-\frac{1}{2}} \qquad (2.70)}$$

Putting 2.70 into 2.65 gives an exact expression for the potential V, although it is not very easy to differentiate. A long way from the dipole, where $r \gg d$, the terms

$$x = \frac{d^2}{r^2} \mp \frac{2d\cos\theta}{r} \tag{2.71}$$

are small ($\ll 1$) and we can use the binomial expansion

$$(1+x)^n = 1 + n\,x + \frac{n(n-1)}{2}\,x^2 + \dots \tag{2.72}$$

with $n = -\frac{1}{2}$ to get

$$(1+x)^{-\frac{1}{2}} \cong 1 - \frac{1}{2}x + \dots \tag{2.73}$$

so that

$$\frac{1}{r_+} - \frac{1}{r_-} = \frac{1}{r}\left[\left(1 - \frac{d^2}{2r^2} + \frac{d\cos\theta}{r} + \dots \right) - \left(1 - \frac{d^2}{2r^2} - \frac{d\cos\theta}{r} + \dots \right) \right] \tag{2.74}$$

therefore

$$\frac{1}{r_+} - \frac{1}{r_-} = \frac{1}{r}\,\frac{2d\cos\theta}{r} \tag{2.75}$$

and

$$\boxed{V_{\text{dipole}} = \frac{q}{4\pi\epsilon_0}\,\frac{2d\cos\theta}{r^2} \quad (r \gg d) \qquad (2.76)}$$

which is the dipole potential at large distances. The quantity $p = 2qd$ is called the **electric dipole moment**.

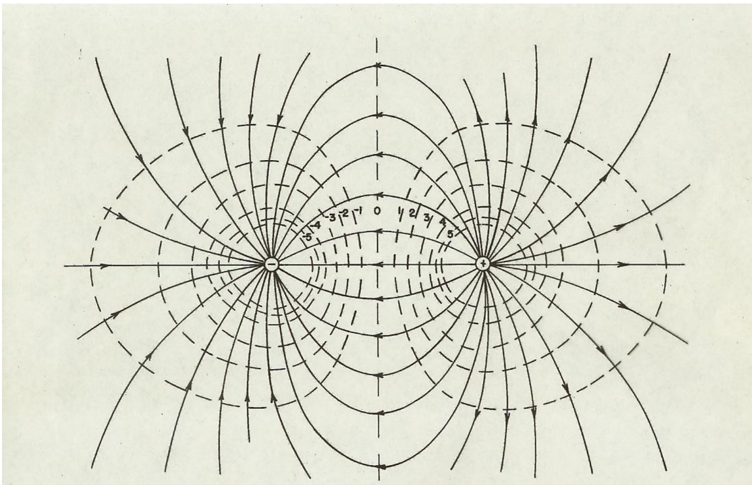


Fig. 24. Field lines and equipotentials for two equal and opposite point charges.

Exercise (the electric quadrupole): consider the charge configuration formed by a charge $-2q$ at the origin and two charges $+q$ at the points $(\pm a, 0, 0)$. Show that the potential V at a distance r large compared with a is approximately given by $V = +qa^2(3\cos^2\theta - 1)/4\pi\epsilon_0 r^3$, where θ is the angle between r and the line through the charges.

2.12 Potential for a continuous charge distribution

For a point charge

$$V = \frac{q}{4\pi\epsilon_0}\,\frac{1}{r}; \qquad \underline{E} = \frac{q}{4\pi\epsilon_0}\,\frac{1}{r^2}\underline{\hat{r}} \tag{2.77}$$

For the case of a continuous distribution of charge, we apply the principle of superposition to a small element of charge dq (see Figure 25)

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\mathrm{d}q}{r} \tag{2.78}$$

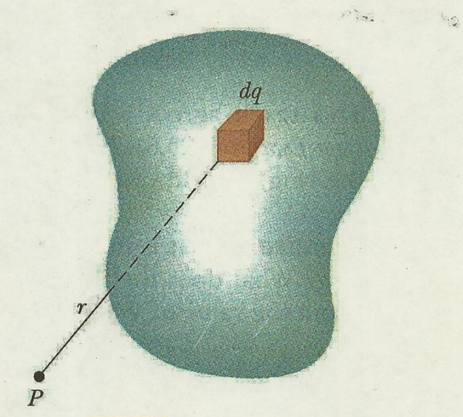


Figure 25. The electric potential at the point P due to a continuous charge distribution can be calculated by dividing the charged body into segments of charge dq and summing the electric potential contributions over all segments.

More explicitly, if the distribution is described by a charge density $\rho(x, y, z)$, then the potential at (x_i, y_i, z_i) is:

$$V(x_i, y_i, z_i) = \frac{1}{4\pi\epsilon_0}\int_{\text{all space containing charge}} \frac{\rho(x_j, y_j, z_j)}{r_{ij}}\mathrm{d}\tau \tag{2.79}$$

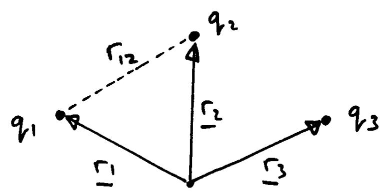
where $\mathrm{d}\tau = \mathrm{d}x\mathrm{d}y\mathrm{d}z$ is a volume element and $r_{ij} = |\underline{r}_{ij}| = |\underline{r}_i - \underline{r}_j|$.

Exercise: Find an expression for the electric potential at a point P located on the perpendicular central axis of a uniformly charged ring of radius a and total charge Q. Find an expression for the magnitude of the electric field at point P.

Exercise: A uniformly charged disk has radius a and surface charge density σ. Find the electric potential along the perpendicular central axis of the disk. By differentiating the electric potential, determine the magnitude of the electric field along the same axis.

2.13 Electrostatic energy: the case of a collection of discrete charges

The electrostatic potential energy U of a system of point charges equals the work W needed to bring the charges from an infinite separation to their final positions. Consider a system of three charges Suppose q_1 was there first. No work is required to place it in



position. The electric potential V_1 due to q_1 at a point $\underline{r}_2$ is

$$V_1(\underline{r}_2) = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r_{12}} \tag{2.80}$$

where $r_{12} = |\underline{r}_1 - \underline{r}_2|$ To bring q_2 from infinity to $\underline{r}_2$ we must do work against the field from q_1. This, from the definition of the potential, is

$$W_2 = q_2\ V_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}} \tag{2.81}$$

To bring up q_3 we do work against the fields due to <u>both</u> q_1 and q_2:

$$W_3 = q_3\ V_1 + q_3\ V_2 = \frac{1}{4\pi\epsilon_0} \left(\frac{q_3\ q_1}{r_{13}} + \frac{q_3\ q_2}{r_{23}} \right) \tag{2.82}$$

Thus the total potential energy of the three charges is

$$U = W_2 + W_3 = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1\ q_2}{r_{12}} + \frac{q_2\ q_3}{r_{23}} + \frac{q_3\ q_1}{r_{31}} \right) \tag{2.83}$$

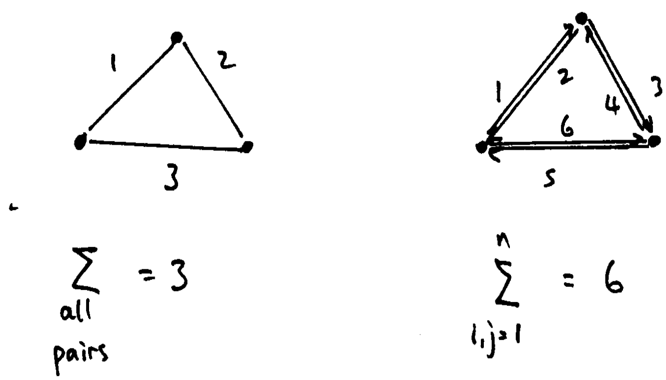
This result can be generalised to a collection of n charges $q_1, q_2, \ldots, q_n$ at positions $\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_n$

$$\boxed{U = \frac{1}{4\pi\epsilon_0} \sum_{\text{all distinct pairs}} \frac{q_i\ q_j}{r_{ij}} \qquad (2.84)}$$

Note that we could also write this as

$$U = \frac{1}{4\pi\epsilon_0}\frac{1}{2}\sum_{i,j=1,i\neq j}^{n}\frac{q_i\ q_j}{r_{ij}} \qquad (2.85)$$

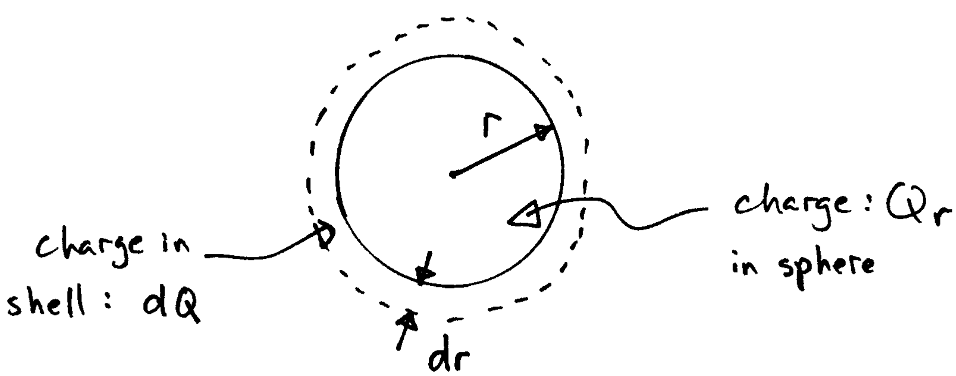
where the factor of $\frac{1}{2}$ is needed as every pair is counted twice in the sum in this case.



2.14 Electrostatic energy: the case of a continuous charge distribution

For continuous distributions the summations of equations 2.84 and 2.85 become integrals. Here we consider only one special case - a uniform sphere of charge of radius a.

To find U, imagine that we assemble the sphere by building up a succession of spherical shells of infinitesimal thickness.



Suppose that the sphere has been partially assembled and currently has a radius r and a charge Q_r. The potential due to the sphere at this stage is

$$V_r = \frac{1}{4\pi\epsilon_0} \frac{Q_r}{r} \quad (2.86)$$

The work done to bring up a further spherical shell of charge dQ is then

$$\mathrm{d}U = \mathrm{d}Q \; V_r = \frac{Q_r \; \mathrm{d}Q}{4\pi\epsilon_0 \; r} \quad (2.87)$$

Suppose that the charge is uniformly distributed with a charge density ρ, then Q_r is given by

$$Q_r = \frac{4}{3} \pi \; r^3 \; \rho \quad (2.88)$$

and the charge in the infintesimal shell is

$$\mathrm{d}Q = 4\pi \; r^2 \; \rho \; \mathrm{d}r \quad (2.89)$$

Substituting these two results in equation 2.87

$$\mathrm{d}U = \frac{1}{4\pi\epsilon_0 r} \left(\frac{4}{3}\pi r^3 \; \rho\right) \left(4\pi r^2 \; \rho\right) \; \mathrm{d}r \quad (2.90)$$

therefore

$$\mathrm{d}U = \frac{4\pi \; \rho^2}{3 \; \epsilon_0} \; r^4 \; \mathrm{d}r \quad (2.91)$$

Integrating from $r = 0$ to $r = a$ gives

$$U = \frac{4\pi \; \rho^2}{3 \; \epsilon_0} \frac{a^5}{5} \quad (2.92)$$

which can be expressed in terms of the total charge on the sphere $Q = \frac{4}{3}\pi \; a^3 \; \rho$ to give

$$U = \frac{3}{5} \frac{Q^2}{4\pi\epsilon_0 \; a} \quad (2.93)$$

Exercise: In the example of the charged sphere of radius a above, consider a sphere of total charge Q whose charge distribution is given by $\rho(r) = Ar$ where A is a constant. By calculating the total charge on the sphere, show that $A = \frac{Q}{\pi a^4}$. Hence show that the total electrostatic energy of the sphere is $\frac{4}{7}\frac{Q^2}{4\pi\epsilon_0 a}$.