9 Maxwell's Equations in differential form

9.1 Summary of Maxwell's equations

We have now seen all four of Maxwell's equations. They are

$$\begin{aligned}
\int_S \underline{E} \cdot \mathrm{d}\underline{a} &= \frac{Q}{\epsilon_0} \quad \text{Gauss' law} && (9.1)\\
\int_S \underline{B} \cdot \mathrm{d}\underline{a} &= 0 \quad \text{Gauss' law in magnetism} && (9.2)\\
\oint \underline{E} \cdot \mathrm{d}\underline{s} &= -\frac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}t} \quad \text{Faraday's law} && (9.3)\\
\oint \underline{B} \cdot \mathrm{d}\underline{s} &= \mu_0 I + \epsilon_0\mu_0 \frac{\mathrm{d}\Phi_\mathrm{E}}{\mathrm{d}t} \quad \text{Ampere} - \text{Maxwell law} && (9.4)
\end{aligned}$$

We have also seen their differential forms:

$$\begin{aligned}
\underline{\nabla} \cdot \underline{E} &= \frac{\rho}{\epsilon_0} \quad \text{Gauss' law} && (9.5)\\
\underline{\nabla} \cdot \underline{B} &= 0 \quad \text{Gauss' law in magnetism} && (9.6)\\
\underline{\nabla} \times \underline{E} &= -\frac{\partial \underline{B}}{\partial t} \quad \text{Faraday's law} && (9.7)\\
\underline{\nabla} \times \underline{B} &= \mu_0 \underline{J} + \epsilon_0\mu_0 \frac{\partial \underline{E}}{\partial t} \quad \text{Ampere} - \text{Maxwell law} && (9.8)
\end{aligned}$$

9.2 Waves in free space

To summarise, Maxwell's equations in differential form in free space (no charges or currents present) are

$$\begin{aligned}
\underline{\nabla} \cdot \underline{E} &= 0 && (9.9)\\
\underline{\nabla} \cdot \underline{B} &= 0 && (9.10)\\
\underline{\nabla} \times \underline{E} &= -\frac{\partial \underline{B}}{\partial t} && (9.11)\\
\underline{\nabla} \times \underline{B} &= \mu_0\epsilon_0 \frac{\partial \underline{E}}{\partial t} && (9.12)
\end{aligned}$$

Now, taking the curl of 9.11 we get

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = -\frac{\partial\left(\underline{\nabla} \times \underline{B}\right)}{\partial t} = -\mu_0\epsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2} \tag{9.13}$$

where we have used equation 9.12.

We also have a general result for $\underline{\nabla} \times (\underline{\nabla} \times \underline{B})$ which gives

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = \underline{\nabla}\,(\underline{\nabla} \cdot \underline{E}) - (\underline{\nabla} \cdot \underline{\nabla})\,\underline{E} \tag{9.14}$$

But $\underline{\nabla} \cdot \underline{E} = 0$ in empty space, so

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = -\nabla^2 \underline{E} \tag{9.15}$$

and hence

$$\boxed{\nabla^2 \underline{E} = \mu_0 \epsilon_0 \frac{\partial^2 \underline{E}}{\partial t^2} \qquad (9.16)}$$

∇^2 is the Laplacian operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

We can, by the same approach, obtain a similar equation for $\underline{B}$

$$\boxed{\nabla^2 \underline{B} = \mu_0 \epsilon_0 \frac{\partial^2 \underline{B}}{\partial t^2} \qquad (9.17)}$$

Equations 9.16 and 9.17 are three-dimensional wave equations. The general 3-D wave equation is

$$\nabla^2 \underline{A} = \frac{1}{v^2} \frac{\partial^2 \underline{A}}{\partial t^2}$$

where v is the speed of propagation of the wave. By comparing with 9.16 and 9.17 we see that the oscillations in $\underline{E}$ and $\underline{B}$ propagate at a speed

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{9.18}$$

which is the speed of light. Equations 9.16 and 9.17 describe waves comprising oscillations in electric and magnetic fields that propagate at the speed of light in a vacuum, in the absence of electric charges or currents.