8 AC Circuits

8.1 Why use alternating current

- The oscillations in an RLC series circuit can be prevented from damping out if an external EMF

$$\mathcal{E} = \mathcal{E}_{\text{max}} \sin \omega t \tag{8.1}$$

  is used to replace the energy dissipated as heat in the resistor.

- Circuits in homes, offices and factories (including countless RLC circuits) receive such an EMF from electricity generating companies: in the UK, $\mathcal{E}_{\text{max}} = 230$ V and the period of oscillation is $T = \frac{2\pi}{\omega} = \frac{1}{50 \text{ Hz}} = \frac{1}{50}$ seconds.

- This EMF delivers an '<u>alternating current</u>' (AC):

$$I = I_{\text{max}} \sin(\omega t - \phi) \tag{8.2}$$

  where ϕ is a phase angle.

- What does this look like at the microscopic level?

- The drift speed of electrons in household wiring is approximately $4 \times 10^{-5}\text{ms}^{-1}$. If we reverse their direction every $\frac{1}{100}$ sec, the electrons can move only about 4×10^{-7}m in a half-cycle, which is about 100 atomic spacings before their motion is reversed. Is this still useful?

- Yes, because if we consider a plane in the conductor a large current can still flow across it if there are enough electrons in motion. Energy can still be dissipated in a resistance, even if the electrons continually reverse their direction of motion.

- AC has other useful features:

  - It is easy to generate
  - It is easy to transform (change voltage)
  - It is easy to use - for example to drive motors, where the oscillation of the current is essential and to deliver energy, in light bulbs for example.

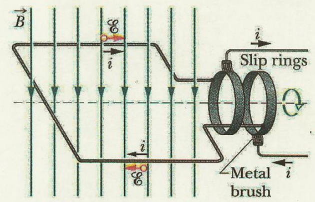


Fig. 89 The basic mechanism of an alternating-current generator is a conducting loop rotated in an external magnetic field. In practice, the alternating emf induced in a coil of many turns of wire is made accessible by means of slip rings attached to the rotating loop. Each ring is connected to one end of the loop wire and is electrically connected to the rest of the generator circuit by a conducting brush against which it slips as the loop (and it) rotates.

8.2 AC generators and transformers

Figure 89 shows a simple model for an AC generator. As the conducting loop is forced to rotate through the field $\underline{B}$, the sinusoidal EMF of equation 8.1 is generated.

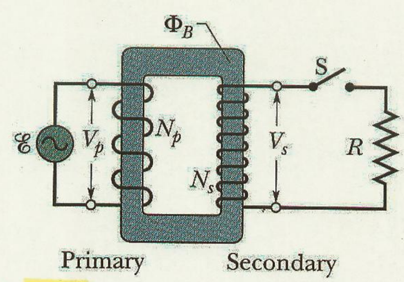


Fig. 90 An ideal transformer (two coils wound on an iron core) in a basic transformer circuit. An ac generator produces current in the coil at the left (the *primary*). The coil at the right (the *secondary*) is connected to the resistive load R when switch S is closed.

Figure 90 shows an ideal transformer. Two coils with different numbers of turns - N_p for the 'primary' coil, N_s for the 'secondary' coil - are wound around an iron core. The primary coil is connected to an AC source of EMF, symbol as shown -

which induces an alternating magnetic flux Φ_B within the iron. Faraday's law gives:

$$\mathcal{E}_p = \Delta V_p = -N_p \frac{\mathrm{d}\Phi_B}{\mathrm{d}t} \tag{8.3}$$

$$\text{and} \quad \mathcal{E}_s = \Delta V_s = -N_s \frac{\mathrm{d}\Phi_B}{\mathrm{d}t} \tag{8.4}$$

for the induced voltage drop across the secondary, so that

$$\boxed{\Delta V_s = \frac{N_s}{N_p}\Delta V_p \qquad (8.5)}$$

- When $N_s > N_p$ the output voltage exceeds the input voltage ('step-up'), and when $N_s < N_p$ the output voltage is less than the input ('step-down').

8.3 Three simple circuits

We will consider, in turn, three simple circuits comprising an AC EMF generator and a load of either a resistor, a capacitor or an inductor. In doing so we will introduce 'phasor diagrams' to help us in more complex circuits where we will want to combine sinusoidal currents and voltages with different phases.

8.3.1 A resistive load

- See Figure 91. Kirchhoff's loop rule must be valid for all t, so

$$\mathcal{E} - \Delta V_R = 0 \tag{8.6}$$

where ΔV_R is the instantaneous voltage across the resistor.

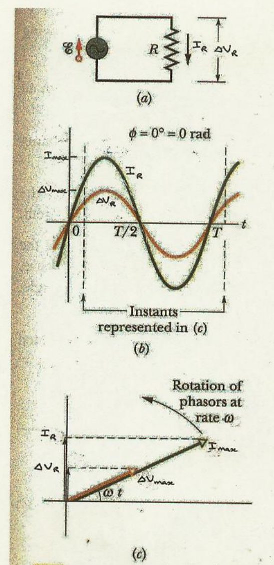


Fig. 91 (a) A resistor is connected across an alternating-current generator. (b) The current I_R and the potential difference ΔV_R across the resistor are plotted on the same graph, both versus time t. They are in phase and complete one cycle in one period T. (c) A phasor diagram shows the same thing as (b).

$$\boxed{\therefore \Delta V_R = \mathcal{E}_{\max} \sin \omega t = \Delta V_{\max} \sin \omega t \qquad (8.7)}$$

since the maximum potential difference across the resistor ΔV_R is the same as that across the source of EMF, $\mathcal{E}_{\max}$. The instantaneous current in the resistor is

$$I_R = \frac{\Delta V_R}{R} = \frac{\Delta V_{\max}}{R} \sin \omega t \tag{8.8}$$

$$\boxed{I_R = I_{\max} \sin \omega t \qquad (8.9)}$$

- Both ΔV_R and I_R vary with $\sin \omega t$ (see Figure 91), they are <u>in phase</u> with each other.

8.3.2 A capacitative load

See Figure 92. Kirchhoff's loop rule again gives

$$\mathcal{E} - \Delta V_C = 0 \tag{8.10}$$

Hence

$$\boxed{\Delta V_C = \Delta V_{\max} \sin \omega t \qquad (8.11)}$$

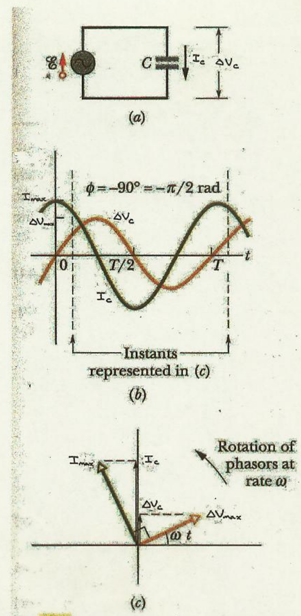


Fig. 92 (a) A capacitor is connected across an alternating-current generator. (b) The current in the capacitor leads the voltage by 90° (= $\pi/2$ rad). (c) A phasor diagram shows the same thing.

Now, from the definition of capacitance ($C = \frac{Q}{\Delta V_C}$), the instantaneous charge on the capacitor is

$$Q = C.\Delta V_C = C.\Delta V_{\max} \sin \omega t \tag{8.12}$$

so the current in the circuit is

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t} = \omega C.\Delta V_{\max} \cos \omega t \tag{8.13}$$

$$\boxed{\therefore I = I_{\max} \sin(\omega t + \frac{\pi}{2}) \qquad (8.14)}$$

We see that the potential difference across the capacitor and the current in the circuit are $\frac{\pi}{2}$ radians out of phase (see Figure 92). The current leads the voltage by $\frac{\pi}{2}$ radians. The maximum current

$$I_{\max} = \omega.C.\Delta V_{\max} \tag{8.15}$$

can be written in terms of an equivalent resistance, called the 'capacitive reactance', X_C,

$$\boxed{I_{\max} = \frac{\Delta V_{\max}}{X_C}, \qquad X_C = \frac{1}{\omega C} \qquad (8.16)}$$

X_C has the units of ohms.

- Note: the phasor diagram is a way to represent the sinusoidal variations in current and voltage in terms of rotating vectors of magnitude $I_{\max}$ and $\Delta V_{\max}$.

8.3.3 An inductive load

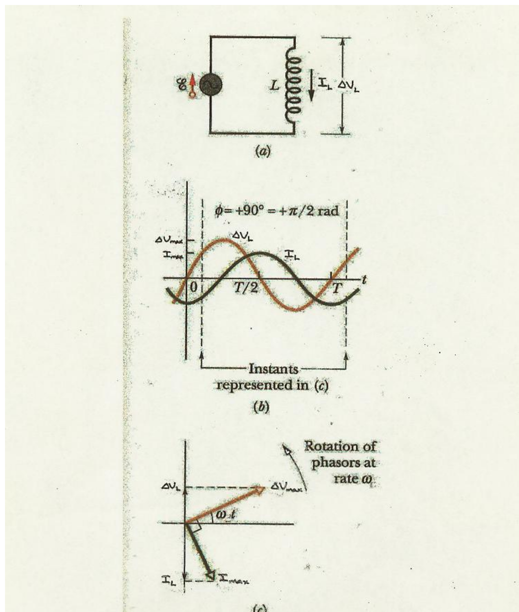


Fig. 93 (a) An inductor is connected across an alternating-current generator. (b) The current in the inductor lags the voltage by 90° (= π/2 rad). (c) A phasor diagram shows the same thing.

See Figure 93. Kirchhoff's rule again gives

$$\boxed{\Delta V_L = \Delta V_{\max} \sin \omega t \qquad (8.17)}$$

From the definition of inductance,

$$\Delta V_L = L \frac{\mathrm{d}I_L}{\mathrm{d}t} \tag{8.18}$$

$$\therefore \mathrm{d}I_L = \frac{\Delta V_{\max}}{L} \sin \omega t \mathrm{d}t \tag{8.19}$$

$$\therefore I_L = \frac{\Delta V_{\max}}{L} \int \sin \omega t \mathrm{d}t = -\frac{\Delta V_{\max}}{\omega L} \cos \omega t \tag{8.20}$$

$$\boxed{I_L = I_{\max}\sin(\omega t - \frac{\pi}{2}) \qquad (8.21)}$$

so ΔV_L and I_L are again $\frac{\pi}{2}$ radians out of phase but now the current lags the voltage by $\frac{\pi}{2}$ (see Figure 93).

- The maximum current, $I_{\max} = \frac{\Delta V_{\max}}{\omega L}$ can be written in terms of an equivalent resistance, the 'inductive reactance', X_L,

$$\boxed{I_{\max} = \frac{\Delta V_{\max}}{X_L}, \qquad X_L = \omega L \qquad (8.22)}$$

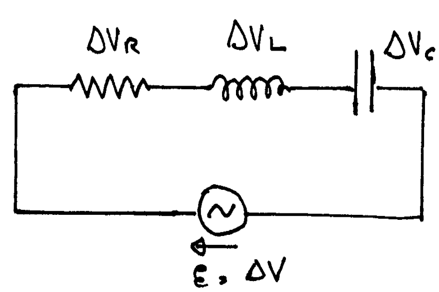
- X_L has units of ohms.

8.4 The RLC circuit

We will now consider AC circuits comprising different combinations of R, L and C in series with an AC source with voltage given by

$$\mathcal{E} = \Delta\mathcal{E}_{\max}\cos\omega t \ . \qquad (8.23)$$

Note that we are using $\cos\omega t$ rather than the $\sin\omega t$ of the previous sections. The reason will shortly become apparent.



As it is a series circuit, the same current flows in all parts of the circuit,

$$I = I_{\max}\cos(\omega t - \phi) \qquad (8.24)$$

and our aim will be to determine $I_{\max}$ and the phase angle ϕ in a given circuit.

It is useful to use complex notations in the calculations. Using the relation

$$\exp(j\omega t) = \cos\omega t + j\sin\omega t \ ,$$

where[1] $j^2 = -1$, we may replace the cosine and sine functions by complex exponentials on the understanding that we are interested in the real or imaginary parts respectively.

We begin by writing the time-variation of the EMF,

$$\boldsymbol{\mathcal{E}} = \mathcal{E}_{\max}\exp(j\omega t) \tag{8.25}$$

where the bold face, $\boldsymbol{\mathcal{E}}$ denotes a complex number and we understand that the real physical EMF is

$$\mathcal{E} = \Re\left(\boldsymbol{\mathcal{E}}\right) \tag{8.26}$$

The current in the circuit will also vary with angular frequency ω so we can write

$$\mathbf{I}(t) = \mathbf{I}_0\exp(j\omega t) \tag{8.27}$$

where the presence of the complex $\mathbf{I}_0$ means that the current and voltage may not vary in phase as we shall see later. The real physical current will be

$$I(t) = \Re(\mathbf{I}_0\exp(j\omega t)) \ . \tag{8.28}$$

Kirchhoff's loop rule is now

$$\boldsymbol{\mathcal{E}} - R\mathbf{I} - L\frac{\mathrm{d}\mathbf{I}}{\mathrm{d}t} - \frac{\mathbf{Q}}{C} = 0 \ . \tag{8.29}$$

Differentiating through with respect to time we have

$$\frac{\mathrm{d}\boldsymbol{\mathcal{E}}}{\mathrm{d}t} = R\frac{\mathrm{d}\mathbf{I}}{dt} + L\frac{\mathrm{d}^2\mathbf{I}}{\mathrm{d}t^2} + \frac{\mathbf{I}}{C} \ . \tag{8.30}$$

Now using equations 8.25 and 8.27, we have

$$j\omega\mathcal{E}_{\max}\exp(j\omega t) = j\omega R\mathbf{I_0}\exp(j\omega t) - \omega^2 L\mathbf{I_0}\exp(j\omega t) + \frac{\mathbf{I_0}}{C}\exp(j\omega t) \ . \tag{8.31}$$

[1] We note that the symbol j rather than i is used in the context of electromagnetism in order to avoid confusion with current

Dividing through by $j\omega$ we find

$$\mathcal{E}_{\max} \exp(j\omega t) = \mathbf{I_0} \exp(j\omega t) \left[R + j \left(\omega L - \frac{1}{\omega C} \right) \right] \ . \tag{8.32}$$

This equation has the form

$$\boldsymbol{\mathcal{E}} = \mathbf{IZ} \tag{8.33}$$

where

$$\mathbf{Z} = R + j \left(\omega L - \frac{1}{\omega C} \right) = R + j(X_L - X_C) \tag{8.34}$$

is the <u>complex impedance</u> of the circuit. The complex impedance can be written in exponential form as

$$\mathbf{Z} = Z \exp i\phi \tag{8.35}$$

where Z is the magnitude of the complex quantity $\mathbf{Z}$

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \tag{8.36}$$

and the angle ϕ is given by

$$\boxed{\phi = \tan^{-1} \frac{X_L - X_C}{R} \qquad (8.37)}$$

The current can be written as

$$\mathbf{I} = \frac{\mathcal{E}_{\max}}{Z} \exp j(\omega t - \phi) \tag{8.38}$$

- The voltage leads the current if $X_L > X_C$ (peaks in current occurr after the peaks in voltage), and lags behind the current if $X_L < X_C$.

- Note that by changing the driving frequency ω of the EMF source we can 'tune' the circuit. For example, at large ω we can be sure that $X_L = \omega L > X_C = \frac{1}{\omega C}$, and that the voltage leads the current.

- At the '<u>resonant frequency</u>', ω_0, we make $X_L = X_C$ so that

$$\omega_0 L = \frac{1}{\omega_0 C} \tag{8.39}$$

and hence

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}} \qquad (8.40)}$$

- At the resonant frequency
  - the impedance equals the resistance, $Z = R$,
  - the voltage is in phase with the current,
  - the current has its maximum value, $I_{\max} = \frac{\Delta V_{\max}}{Z} = \frac{\Delta V_{\max}}{R}$

8.5 Power in the RLC series circuit

- The energy source in the circuit is the AC EMF generator. This energy is
  - stored in the electric field of the capacitor,
  - stored in the magnetic field of the inductor, and
  - dissipated as thermal energy in the resistor.

- In the steady state, the average energy stored in the capacitor and the inductor is constant, so the net transfer of energy is from the source of EMF to the resistor. The rate of energy transfer, ie the power, is then

$$P = I\Delta V = I_{\max}\cos(\omega t - \phi)\Delta V_{\max}\cos\omega t \qquad (8.41)$$

This is the instantaneous power delivered by the EMF to the resistor, as a function of time.

- The average power delivered can be found by first noting that:

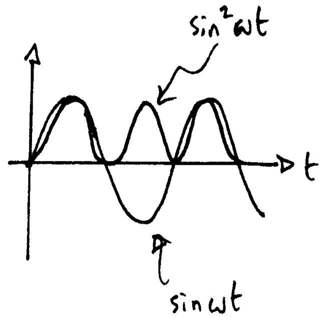
$$\cos(\omega t - \phi) = \cos\omega t\cos\phi + \sin\omega t\sin\phi \qquad (8.42)$$

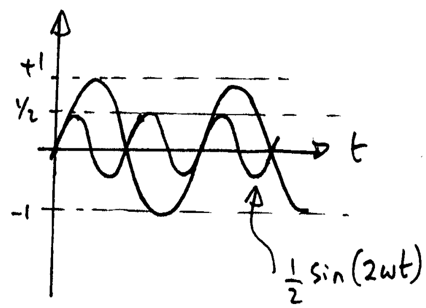
so that equation 8.41 becomes

$$P = I_{\max}\Delta V_{\max}\left[\cos^2\omega t\cos\phi + \sin\omega t\cos\omega t\sin\phi\right] \qquad (8.43)$$

and secondly, noting that the time average of $\sin^2\omega t$ is $\frac{1}{2}$, while the time average of $\sin\omega t\cos\omega t = \frac{1}{2}\sin(2\omega t)$ is zero. The average power is therefore

$$\boxed{P_{\mathrm{av}} = \frac{1}{2}I_{\max}\Delta V_{\max}\cos\phi \qquad (8.44)}$$





8.6 Average power in terms of rms current

- Equation 8.44 can be written in terms of the 'root mean square' (rms) values for current and voltage:

$$
\begin{aligned}
I_{\text{rms}} &= \sqrt{\text{average value of } I^2} && (8.45)\\
&= \sqrt{\text{average value of } I_{\text{max}}^2 \sin^2(\omega t - \phi)} && (8.46)\\
&= \sqrt{\frac{1}{2} I_{\text{max}}^2} && (8.47)
\end{aligned}
$$

$$\boxed{\therefore I_{\text{rms}} = \frac{1}{\sqrt{2}} I_{\text{max}} \qquad (8.48)}$$

and similarly,

$$\boxed{\Delta V_{\text{rms}} = \frac{1}{\sqrt{2}} \Delta V_{\text{max}} \qquad (8.49)}$$

so that

$$\boxed{\therefore P_{\text{av}} = I_{\text{rms}} \Delta V_{\text{rms}} \cos\phi \qquad (8.50)}$$

where $\cos\phi$ is called the 'power factor'. Now, in an RLC circuit:

$$
\begin{aligned}
I_{max} &= V_{max}/|Z| && (8.51)\\
\tan^{-1} &= (X_L - X_C)/R && (8.52)
\end{aligned}
$$

which implies

$$\cos\phi = \frac{I_{\text{max}}R}{\Delta V_{\text{max}}} = \frac{I_{\text{rms}}R}{\Delta V_{\text{rms}}} \tag{8.53}$$

which, when substituted in equation 8.50 gives

$$\boxed{P_{\text{av}} = I_{\text{rms}}^2 R \qquad (8.54)}$$

which restates our original assumption that, in the ideal RLC circuit there is no power loss in either the inductor or the capacitor.

8.7 Resonance in the RLC series circuit

- As already discussed, the RLC circuit is said to be in resonance when the current is at its maximum value. For the rms current

$$I_{\text{rms}} = \frac{\Delta V_{\text{rms}}}{Z} \tag{8.55}$$

this occurs when the impedance

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \tag{8.56}$$

is a minimum, which is when $X_L = X_C$, corresponding to $\omega = \omega_0$, the resonance frequency where

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{8.57}$$

- The average power shows a resonance peak since

$$P_{\text{av}} = I_{\text{rms}}^2 R = \frac{\Delta V_{\text{rms}}^2 R}{Z^2} \tag{8.58}$$

which can be written as

$$\boxed{P_{\text{av}} = \frac{(\Delta V_{\text{rms}})^2 R\omega^2}{R^2\omega^2 + L^2(\omega^2 - \omega_0^2)^2} \qquad (8.59)}$$

This expression shows that at resonance, $\omega = \omega_0$,

$$P_{\text{av}} = \frac{(\Delta V_{\text{rms}})^2}{R} \tag{8.60}$$

and is at a <u>maximum</u>.

- The sharpness of the $P_{\mathrm{av}}(\omega)$ curve is described by the 'quality factor', Q,

$$Q = \frac{\omega_0}{\Delta\omega} \tag{8.61}$$

where $\Delta\omega$ is the full width at half-maximum of the curve. A high-Q circuit responds only to a narrow range of frequencies.

- The receiving circuit of a radio is an application of a resonant circuit. 'Tuning' the radio involves varying a capacitor, which alters the circuit's resonant frequency to match that of the incoming electromagnetic wave.

8.8 Power transmission

- To get power from a generating station (eg a hydroelectric plant) to household supplies, the EMF is passed along large conducting cables called 'transmission' lines. To minimise power losses in the transmission lines ($P = I^2R$), low resistance metals are used (eg copper), coupled with relatively small currents (about 100 A). To keep the transmitted power high, high voltages are used (about 440 kV).