7 Electromagnetic induction

7.1 Magnetic flux

We define the magnetic flux, Φ_B, of the magnetic field $\underline{B}$ through a surface S as the scalar given by

$$\Phi_B = \int_S \underline{B} \cdot d\underline{a} \tag{7.1}$$

The SI unit for magnetic flux is the weber, where one weber = one tesla metre squared.

Example: The magnetic flux through a plane of area A placed in a uniform field $\underline{B}$ is

$$\Phi_B = BA\cos\theta \tag{7.2}$$

where θ is the angle between $\underline{B}$ and the normal to the plane surface.

Example: Figure 74 shows a rectangular loop beside a current carrying wire. What is Φ_B through the planar surface bounded by the loop?

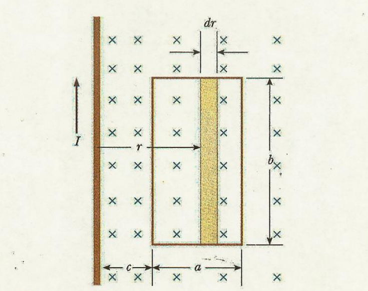


Figure 74 The magnetic field due to the wire carrying a current I is not uniform over the rectangular loop.

We know that

$$B = \frac{\mu_0 I}{2\pi r} \tag{7.3}$$

and that $\underline{B}$ is directed perpendicular to and into the plane of the loop. Therefore:

$$\Phi_B = \int_S \underline{B} \cdot d\underline{a} = \int \frac{\mu_0 I}{2\pi r} da \tag{7.4}$$

The area element $da = b dr$ (see Figure 74) so

$$\Phi_B = \frac{\mu_0 I b}{2\pi} \int_c^{a+c} \frac{dr}{r} = \frac{\mu_0 I b}{2\pi} \log\left(\frac{a+c}{c}\right) = \frac{\mu_0 I b}{2\pi} \log(1 + \frac{a}{c}) \tag{7.5}$$

7.2 Gauss' law for magnetism

Our aim is now to determine div$\underline{B}$. We start by calculating div(d$\underline{B}$) where, as we saw in the previous chapter, d$\underline{B}$ is given by

$$d\underline{B} = \frac{\mu_o I}{4\pi}\frac{d\underline{s}\times\underline{r}}{r^3} \ . \tag{7.6}$$

In calculating div$\underline{B}$, d$\underline{s}$ is a constant vector. We apply the general relation

$$\underline{\nabla}\cdot(\underline{P}\times\underline{Q}) = \underline{Q}\cdot(\underline{\nabla}\times\underline{P}) - \underline{P}\cdot(\underline{\nabla}\times\underline{Q}) \ . \tag{7.7}$$

As d$\underline{s}$ is constant, we have

$$\nabla\cdot\left(\frac{d\underline{s}\times\underline{r}}{r^3}\right) = -\mathrm{d}\underline{s}\cdot\left(\underline{\nabla}\times\frac{\underline{r}}{r^3}\right) \tag{7.8}$$

Therefore we are left with calculating $\underline{\nabla}\times(\underline{r}/r^3)$. By writing the term $(\underline{r}/r^3)$ as

$$\underline{r}/r^3 = -\underline{\nabla}\left(\frac{1}{r}\right)$$

we deduce

$$\underline{\nabla}\times(\underline{r}/r^3) = -\underline{\nabla}\times\underline{\nabla}\frac{1}{r} = 0$$

as $\underline{\nabla}\times\underline{\nabla}f = 0$ for any f.

We therefore conclude that div d$\underline{B} = 0$. If this is true for every contribution d$\underline{B}$, it must also be true for the vector $\underline{B}$ itself. We can therefore write

$$\boxed{\mathrm{div}\underline{B} = 0 \ . \qquad (7.9)}$$

This is Gauss' law for magnetism.

We notice that Eq. 7.9 implies that

$$\Phi_B = \int_S \underline{B}\cdot d\underline{a} = \int_V \mathrm{div}\underline{B}\ \mathrm{d}V = 0 \tag{7.10}$$

for any <u>closed</u> surface S. This means that for a given closed surface S around a volume V as many lines enter the volume V as leave it. In other words, **there are no 'magnetic charges' or 'magnetic poles'** on which magnetic field lines end (or start from), and magnetic field lines are continuos.

7.3 Ampère-Maxwell law for time-varying fields

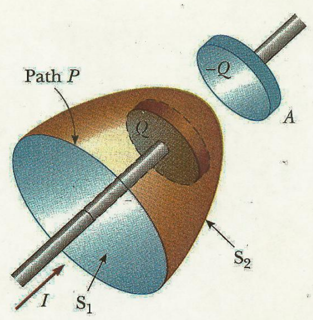


Figure 75 Two surfaces S_1 and S_2 near the plate of a capacitor are bounded by the same path P. The conduction current in the wire passes only through S_1. This leads to a contradiction in Ampère's law that is resolved only if one postulates a displacement current through S_2.

So far we have considered the magnetostatic situation, with steady currents and $\underline{E}$ and $\underline{B}$ unchanged with time. But what if $\underline{E}$, for example varies with time? Consider a charging capacitor (Figure 75). A conduction current I flows to the positive plate, but there is no current between the plates. Consider two surfaces S_1 and S_2, both bounded by the path P. Ampère's law states

$$\oint \underline{B} \cdot \mathrm{d}\underline{s} = \mu_0 I \tag{7.11}$$

where I is the current flowing through any surface bounded by P. For S_1, the RHS of 7.11 is $\mu_0 I$ while for S_2 the RHS is zero. Maxwell concluded that, in the presence of time-varying electric fields, Ampère's law is incomplete. Maxwell postulated a 'displacement current', defined as

$$I_\mathrm{d} = \epsilon_0 \frac{\mathrm{d}\Phi_\mathrm{E}}{\mathrm{d}t} \tag{7.12}$$

where Φ_E is the electric flux and revised Ampère's law to

$$\oint \underline{B} \cdot \mathrm{d}\underline{s} = \mu_0 I + \mu_0 \epsilon_0 \frac{\mathrm{d}\Phi_\mathrm{E}}{\mathrm{d}t} \tag{7.13}$$

This is known as the Ampère-Maxwell law.

For the capacitor in Figure 75, assuming a uniform field $E = \frac{Q}{\epsilon_0 A}$ between the plates, the electric flux through S_2 is

$$\Phi_\mathrm{E} = EA = \frac{Q}{\epsilon_0} \tag{7.14}$$

and the displacement current is then

$$I_d = \epsilon_0 \frac{\mathrm{d}\Phi_{\mathrm{E}}}{\mathrm{d}t} = \frac{\mathrm{d}Q}{\mathrm{d}t} \tag{7.15}$$

which is precisely the same as the conduction current $I = \frac{\mathrm{d}Q}{\mathrm{d}t}$ through S_1. Thus with the Ampère-Maxwell law, the same result is obtained with S_2 as with S_1. Note that the major new concept in equation 7.13 is that **magnetic fields $\underline{B}$ can be produced both by conduction currents <u>and</u> by time-varying electric fields**.

7.4 Faraday's law of electromagnetic induction

We now turn our attention to electric fields produced by changing magnetic fields.

The experiments of Faraday, carried out in 1831, showed that **a transient current occurred in a closed circuit when the magnetic flux through the circuit was changed**. The change in magnetic flux can be produced in various way. In the experiment of Figure 76, a pulse of current is seen in a loop of wire when a magnet is moved toward or away from it.

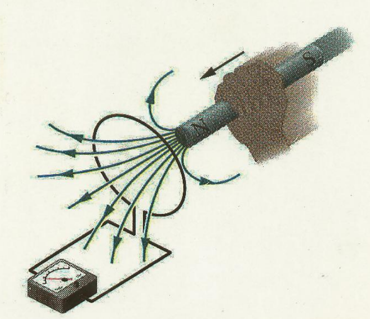


Fig. 76 A current meter registers a current in the wire loop when the magnet moving with respect to the loop.

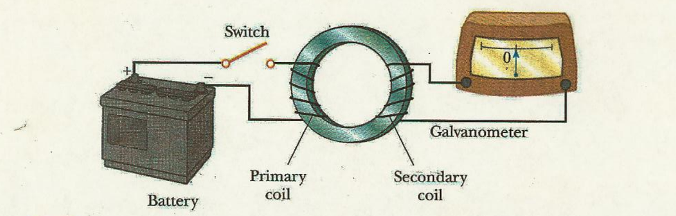


Figure 77 Faraday's experiment. When the switch in the primary circuit is closed, the galvanometer in the secondary circuit deflects momentarily. The emf induced in the secondary circuit is caused by the changing magnetic field through the secondary coil.

In the experiment of Figure 77 a current pulse is induced in the 'secondary' coil when the switch in the 'primary' circuit is opened or closed.

Faraday's conclusions were:

- when the magnetic flux through a circuit is changing, an electromotive force is induced in the circuit;
- the magnitude of this e.m.f. is proportional to the rate of change of the flux.

These results are summarized by the Faraday's law of induction

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}t} \qquad (7.16)$$

where

$$\Phi_\mathrm{B} = \int_S \underline{B} \cdot \mathrm{d}\underline{a} \qquad (7.17)$$

is the magnetic flux through the circuit in which the EMF, $\mathcal{E}$ is induced.

Example: EMF in a conducting bar on rails

Consider Figure 79: a circuit comprising a conducting bar moving along two parallel conducting rails with constant velocity $\underline{v}$ (eg the axle of a train). In the presence of a uniform $\underline{B}$ into the plane, what EMF is induced in the bar? The magnetic flux through the circuit is

$$\Phi_\mathrm{B} = \int_S \underline{B} \cdot \mathrm{d}\underline{a} = B.l.x$$

Therefore using Faraday's law,

$$\mathcal{E} = -\frac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}t} = -B.l.\frac{\mathrm{d}x}{\mathrm{d}t} = -Blv \qquad (7.18)$$

If the resistance of the bar and rails are small compared to the overall resistance R of the circuit, we find the induced current in the circuit to be

$$I = \frac{|\mathcal{E}|}{R} = \frac{Blv}{R} \qquad (7.19)$$

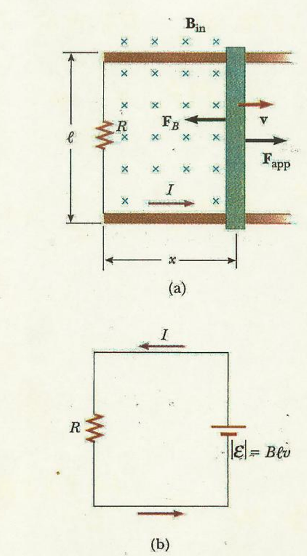
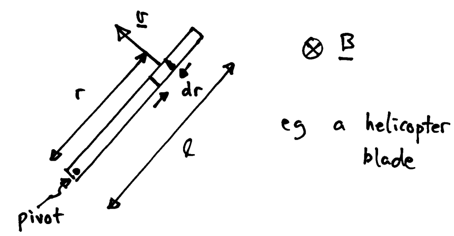


Figure 79 (a) A conducting bar sliding with a velocity **v** along two conducting rails under the action of an applied force $\mathbf{F}_{app}$. The magnetic force $\mathbf{F}_B$ opposes the motion, and a counterclockwise current I is induced in the loop. (b) The equivalent circuit diagram for the setup shown in part (a).

Example: EMF in a rotating bar

Consider a conducting bar rotating with angular speed $\omega = \frac{v}{r}$ in a uniform magnetic field $\underline{B}$ perpendicular to the plane of rotation.
A segment of the bar, of length dr, generates an EMF, which from 7.18 is



$$d\mathcal{E} = -B.v.\mathrm{d}r = -B.\omega.r.\mathrm{d}r \qquad (7.20)$$

so the total EMF across the whole bar is

$$\boxed{\mathcal{E} = -B\omega \int_0^l r\mathrm{d}r = -\frac{1}{2}B\omega l^2 \qquad (7.21)}$$

7.5 Lenz's law

Soon after Faraday proposed his law, Heinrich Lenz devised a rule for finding the direction of the induced current in a loop.

"An induced current has a direction such that the magnetic field due to the current opposes the change in the magnetic flux that induces the current"

As the EMF is in the same direction as the induced current, this corresponds to the minus sign in Faraday's law.

7.6 Self-inductance

Faraday's law tells us that any circuit that carries a time-varying current (and associated time-varying magnetic flux Φ_B) has induced in it an EMF $\mathcal{E}_L$ that opposes the source EMF. This effect is called self-induction, and $\mathcal{E}_L$ is called the self-induced or 'back' EMF

$$\mathcal{E}_L = -\frac{d\Phi_B}{dt} \qquad (7.22)$$

If we think now about immobile circuits (ie no moving parts), in which

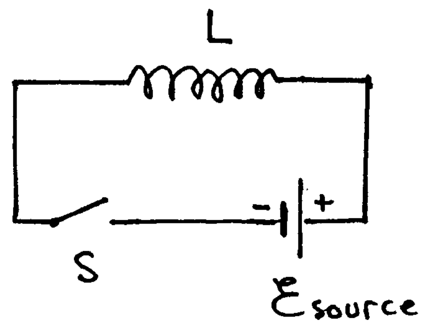
$$\frac{d\Phi_B}{dt} \propto \frac{dB}{dt} \propto \frac{dI}{dt} \qquad (7.23)$$

where I is the source current, we can write 7.22 as

$$\mathcal{E}_L = -L\frac{dI}{dt} \qquad (7.24)$$

where L is a constant, known as the inductance of the circuit.

The archetypal source of induction (or inductor) in a circuit is the ideal solenoid. Let's consider a circuit containing one



when S is closed, current starts to flow. At any given instant the field in the solenoid is

$$B = \frac{\mu_0 N I}{l} \tag{7.25}$$

where $\frac{N}{l}$ is the number of turns per unit length. The magnetic flux through each turn is

$$\Phi_{\mathrm{B}}^{\text{each turn}} = B.A = \frac{\mu_0 N A I}{l} \tag{7.26}$$

where A is the area of the loop. The total flux through the solenoid is

$$\Phi_{\mathrm{B}}^{\text{total}} = N.\Phi_{\mathrm{B}}^{\text{each turn}} = \frac{\mu_0 N^2 A I}{l} \tag{7.27}$$

and hence the induced EMF is

$$\mathcal{E}_{\mathrm{L}} = -\frac{\mathrm{d}\Phi_{\mathrm{B}}}{\mathrm{d}t} = -\frac{\mu_0 N^2 A}{l}\frac{\mathrm{d}I}{\mathrm{d}t} \tag{7.28}$$

so, by comparison with 7.24

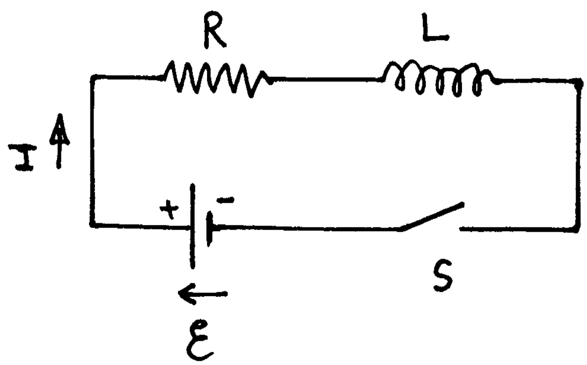
$$\boxed{L = \frac{\mu_0 N^2 A}{l} \qquad (7.29)}$$

which depends only on the geometry of the solenoid.

The SI unit of inductance is the henry. One henry (H) = one volt-second per ampere: $1 \text{ H} = 1 \text{ VsA}^{-1}$.

7.7 RL circuits

An inductor in a circuit opposes changes in the current through that circuit. Consider the 'RL circuit': a resistor and an inductor in series.



When S is closed the back EMF $\mathcal{E}_{\mathrm{L}}$ is induced in the inductor. Applying Kirchhoff's loop rule to the circuit

$$\mathcal{E} - IR - L\frac{\mathrm{d}I}{\mathrm{d}t} = 0 \tag{7.30}$$

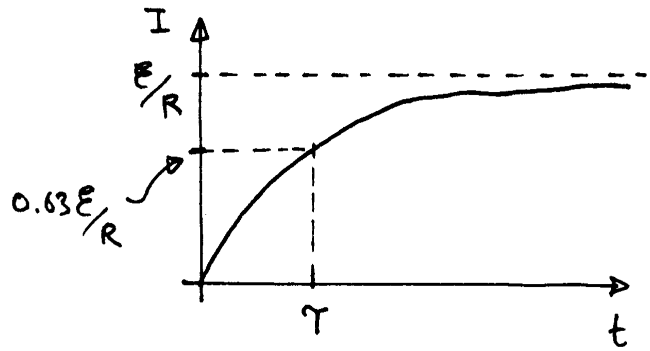
which is a differential equation in $I(t)$, whose solution is

$$I = \frac{\mathcal{E}}{R}(1 - \mathrm{e}^{-\frac{t}{\tau}}) \tag{7.31}$$

where τ is the time-constant of the RL circuit, given by

$$\tau = \frac{L}{R} \; . \tag{7.32}$$

From the graph of 7.31 we find that $I \to \frac{\mathcal{E}}{R}$ as $t \to \infty$, and that at $t = \tau, I = (1 - \frac{1}{\mathrm{e}}).\frac{\mathcal{E}}{R} = 0.63\frac{\mathcal{E}}{R}$. We notice that with no inductor in the circuit the current would have gone to $\frac{\mathcal{E}}{R}$ almost instantaneously.



7.8 Energy stored in a magnetic field

If we multiply 7.30 by I and rearrange it we can obtain

$$\mathcal{E}I = I^2R + LI\frac{\mathrm{d}I}{\mathrm{d}t} \tag{7.33}$$

All the terms in this equation have units of power or energy per unit time. Thus we can read the equation as

- The rate of energy supply by the battery, $\mathcal{E}I$
- equals the rate of energy dissipation in the resistor, I^2R
- plus the rate of energy storage in the inductor, $LI\frac{\mathrm{d}I}{\mathrm{d}t}$.
- We can use this last term to define the energy U stored in the inductor at any time:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = LI\frac{\mathrm{d}I}{\mathrm{d}t} \tag{7.34}$$

Therefore

$$U = \int_0^U \mathrm{d}U = \int_0^I L.I\mathrm{d}I \tag{7.35}$$

$$\boxed{U = \frac{1}{2}LI^2 \qquad (7.36)}$$

- Note the similarity of form to the energy stored in the electric field of a capacitor, $U = \frac{1}{2}\frac{Q^2}{C}$.

- In the case of an inductor which is an ideal solenoid, we have $L = \frac{\mu_0 N^2 A}{l}$, and the magnetic field occupies a volume $A.l$, so that we can write an expression for the energy density of the field

$$u = \frac{U}{A.l} = \frac{\frac{1}{2}\frac{\mu_0 N^2 A}{l} I^2}{A.l} = \frac{1}{2}\frac{\mu_0 N^2 I^2}{l^2} \tag{7.37}$$

But we know that, for a solenoid, $B = \frac{\mu_0 N I}{l}$, so we can write

$$u = \frac{B^2}{2\mu_0} \tag{7.38}$$

This result is generally applicable. Note again the similarity of form compared to the energy density of an electric field $u = \frac{1}{2}\epsilon_0 E^2$.

7.9 Mutual inductance

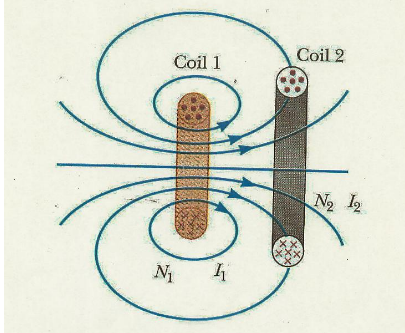


Figure 86 A cross-sectional view of two adjacent coils. A current in coil 1 sets up a magnetic flux, part of which passes through coil 2.

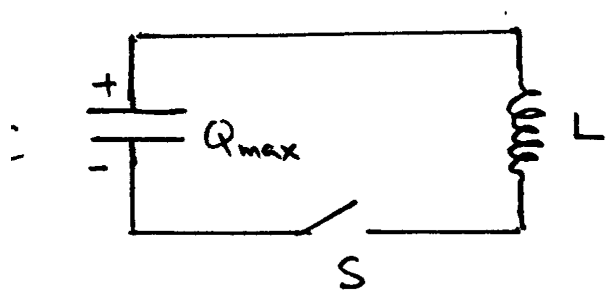
- If two coils are placed next to each other, as in Figure 86, the time-varying current in one produces a time-varying Φ_B in the other, and hence an induced EMF.
- It can be shown that

$$\mathcal{E}_1 = -M\frac{dI_2}{dt}, \quad \mathcal{E}_2 = -M\frac{dI_1}{dt} \tag{7.39}$$

where M is the mutual unductance.

7.10 Oscillations in an LC circuit

We have three circuit elements: resistance, R, capacitance, C, inductance, L. We have previously considered the RC and RL series circuits, now we look at the LC circuit:



Suppose that the capacitor is fully charged to a charge of Q_{max}, and that at time $t = 0$, the switch S is closed. So long as there is no resistance in the circuit, oscillations are set up, see Figure 87.

- At any time t the energy stored in the electric field of the capacitor is

$$U_E = \frac{Q^2}{2C} \tag{7.40}$$

and the energy stored in the magnetic field of the inductor is

$$U_B = \frac{1}{2}LI^2 \tag{7.41}$$

- The total energy of the circuit, $U = U_E + U_B$, remains constant with time, so $\frac{\mathrm{d}U}{\mathrm{d}t} = 0$. So

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{Q^2}{2C} + \frac{1}{2}LI^2\right) \tag{7.42}$$

$$= \frac{Q}{C}\frac{\mathrm{d}Q}{\mathrm{d}t} + LI\frac{\mathrm{d}I}{\mathrm{d}t} \tag{7.43}$$

$$= 0 \tag{7.44}$$

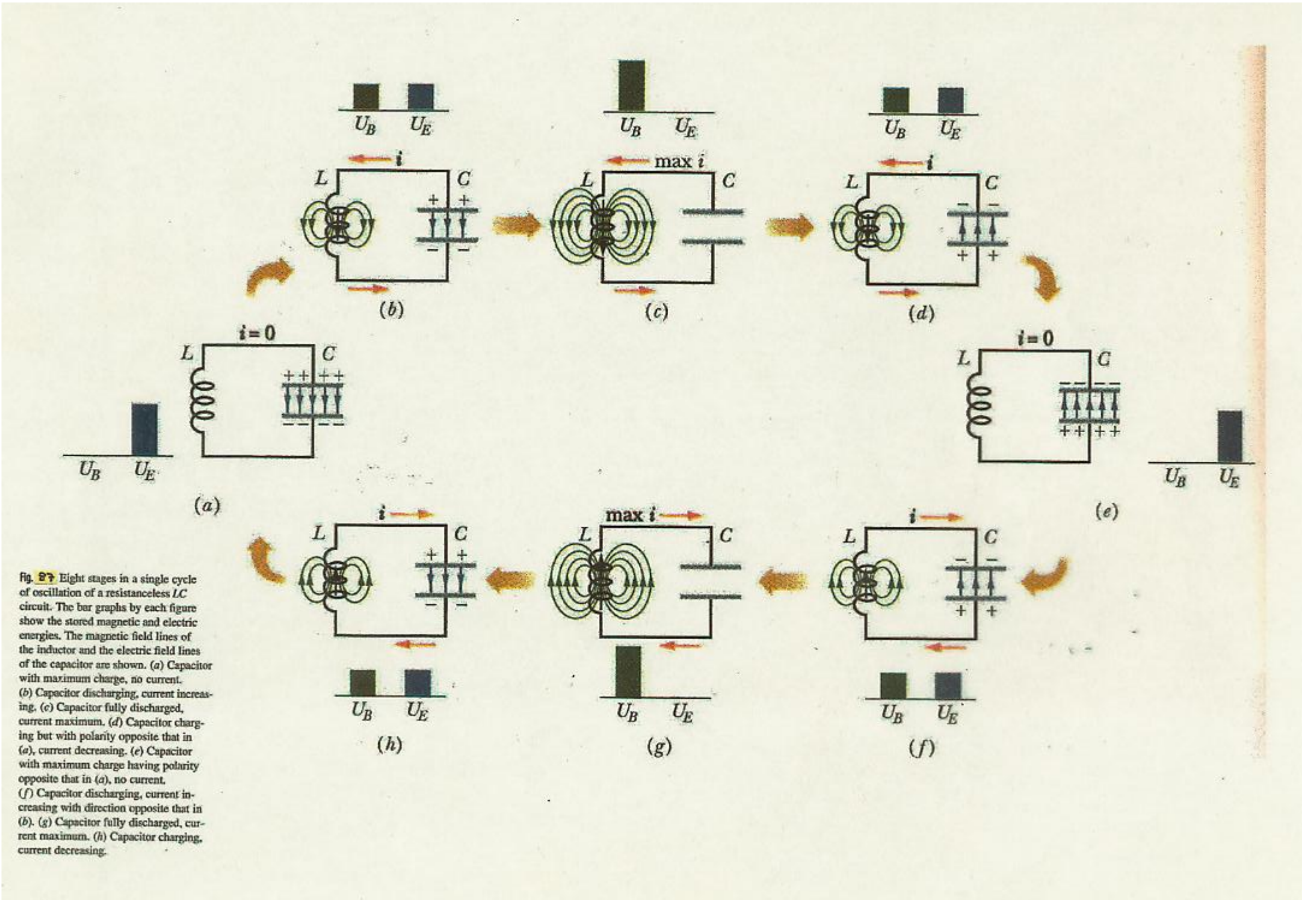


Fig. [illegible] Eight stages in a single cycle of oscillation of a resistanceless LC circuit. The bar graphs by each figure show the stored magnetic and electric energies. The magnetic field lines of the inductor and the electric field lines of the capacitor are shown. (*a*) Capacitor with maximum charge, no current. (*b*) Capacitor discharging, current increasing. (*c*) Capacitor fully discharged, current maximum. (*d*) Capacitor charging but with polarity opposite that in (*a*), current decreasing. (*e*) Capacitor with maximum charge having polarity opposite that in (*a*), no current. (*f*) Capacitor discharging, current increasing with direction opposite that in (*b*). (*g*) Capacitor fully discharged, current maximum. (*h*) Capacitor charging, current decreasing.

But,

$$I = \frac{\mathrm{d}I}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}^2Q}{\mathrm{d}t^2} \tag{7.45}$$

$$\therefore \frac{Q}{C}\frac{\mathrm{d}Q}{\mathrm{d}t} + L\frac{\mathrm{d}Q}{\mathrm{d}t}\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} = 0 \tag{7.46}$$

$$\therefore \frac{\mathrm{d}^2Q}{\mathrm{d}t^2} = -\frac{Q}{LC} \tag{7.47}$$

- Note that this differential equation has the same form as that governing simple harmonic motion. If we define

$$\omega = \frac{1}{\sqrt{LC}} \tag{7.48}$$

equation 7.47 becomes

$$\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} = -\omega^2 Q \tag{7.49}$$

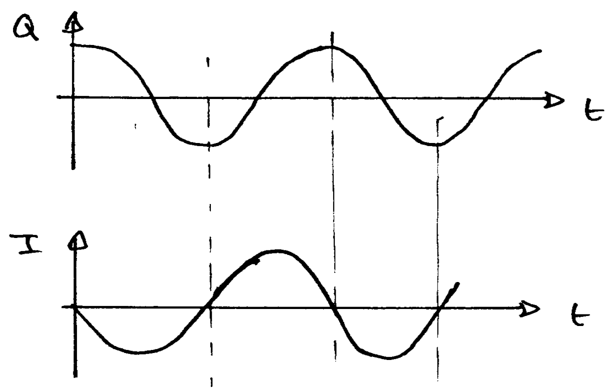
which has solutions

$$Q = A\cos(\omega t + \phi) \tag{7.50}$$

where ϕ is a phase angle and A is a constant.

- In our circuit, the charge is at a maximum, $Q_{\max}$ at $t = 0$, so $A = Q_{\max}$, and the solution is simply
$$Q = Q_{\max} \cos(\omega t) \tag{7.51}$$
and the current is therefore
$$I = \frac{dQ}{dt} = -\omega Q_{\max} \sin(\omega t) \tag{7.52}$$
or
$$I = I_{\max} \sin(\omega t) \tag{7.53}$$

- Graphically, the variations of Q and I are



- We can verify that the total energy in the circuit is constant, since
$$U = \frac{Q^2}{2C} + \frac{1}{2}LI^2 \tag{7.54}$$
$$= \frac{Q_{\max}^2}{2C}\cos^2(\omega t) + \frac{1}{2}LI_{\max}^2 \sin^2(\omega t) \tag{7.55}$$
But since
$$I_{\max}^2 = \omega^2 Q_{\max}^2 = \frac{Q_{\max}^2}{LC} \tag{7.56}$$
so
$$U = \frac{Q_{\max}^2}{2C}(\cos^2 \omega t + \sin^2 \omega t) = \frac{Q_{\max}^2}{2C} \tag{7.57}$$

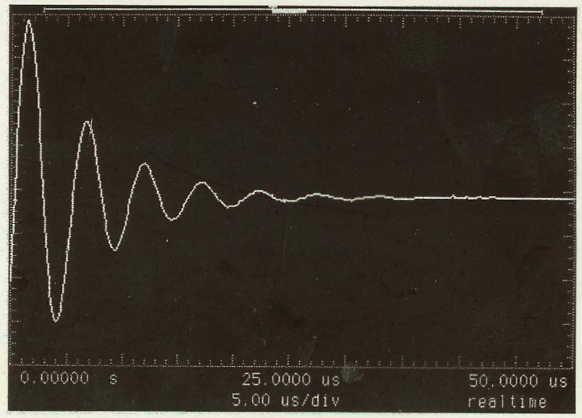
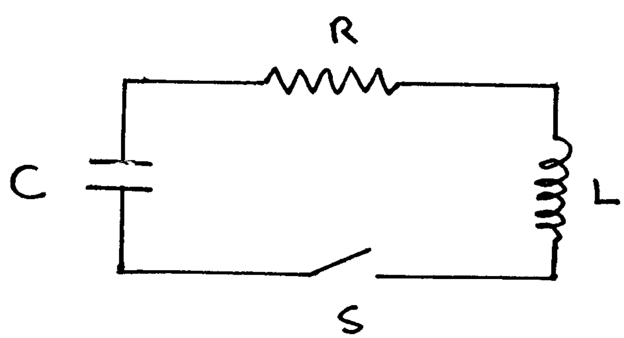


Fig. 88 An oscilloscope trace showing how the oscillations in an *RLC* circuit actually die away because energy is dissipated in the resistor as thermal energy.

7.11 Damped oscillations in an RLC circuit

Adding resistance to an LC circuit is like adding friction to a simple harmonic oscillator, energy is lost and the oscillations are damped down. (see Figure 88).

In such an RLC series circuit:



the rate of energy loss as heat in the resistor is

$$\frac{\mathrm{d}U}{\mathrm{d}t} = -I^2 R \tag{7.58}$$

which must be equal to the rate of change of energy in the other elements of the circuit. We saw above that this has the value

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \frac{Q}{C}\frac{\mathrm{d}Q}{\mathrm{d}t} + LI\frac{\mathrm{d}I}{\mathrm{d}t} \tag{7.59}$$

Equating 7.58 and 7.59 we have

$$\frac{Q}{C}\frac{\mathrm{d}Q}{\mathrm{d}t} + LI\frac{\mathrm{d}I}{\mathrm{d}t} = -I^2R \tag{7.60}$$

or

$$LI\frac{\mathrm{d}I}{\mathrm{d}t} + I^2R + \frac{Q}{C}\frac{\mathrm{d}Q}{\mathrm{d}t} = 0 \tag{7.61}$$

or, in terms of Q,

$$L\frac{\mathrm{d}Q}{\mathrm{d}t}\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + \left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right)^2 R + \frac{Q}{C}\frac{\mathrm{d}Q}{\mathrm{d}t} = 0 \tag{7.62}$$

and, dividing through by $\frac{\mathrm{d}Q}{\mathrm{d}t}$,

$$L\frac{\mathrm{d}^2Q}{\mathrm{d}t^2} + R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} = 0 \tag{7.63}$$

- The solutions to this equation depend on the relative sizes of R, L and C. If R is small, the circuit is 'lightly damped' and the solution is

$$Q = Q_{\mathrm{max}}\mathrm{e}^{-\frac{Rt}{2L}}\cos(\omega_{\mathrm{d}}t), \tag{7.64}$$

where

$$\omega_{\mathrm{d}} = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \tag{7.65}$$

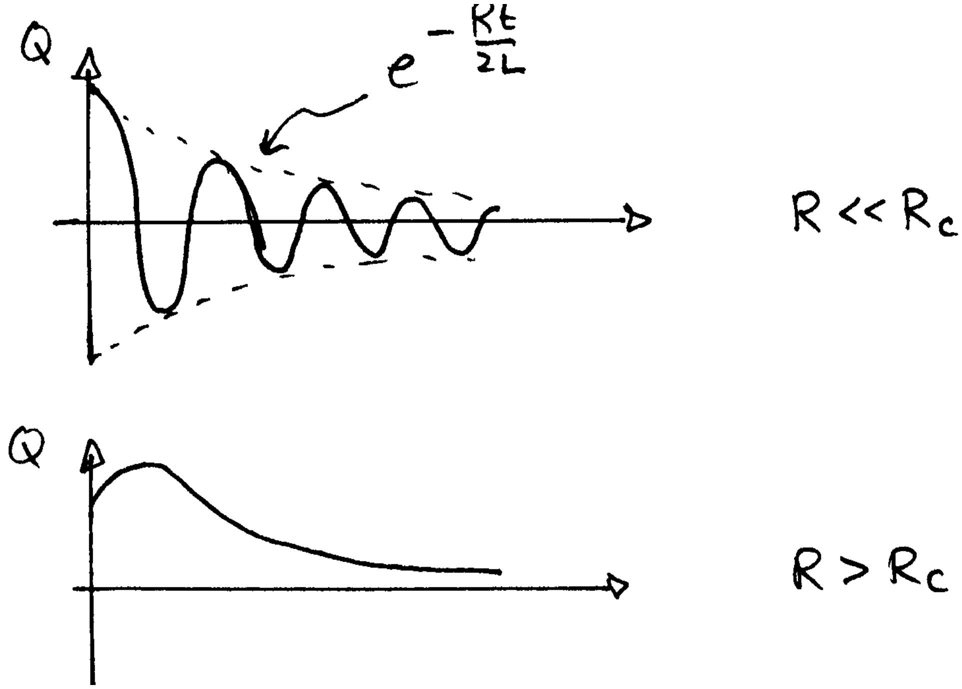
.

as in Figure 88. If R exceeds a critical value R_{c}, given by

$$R_{\mathrm{c}} = \sqrt{\frac{4L}{C}} \tag{7.66}$$

the circuit is said to be 'overdamped' and no oscillations occur.

Graphically,

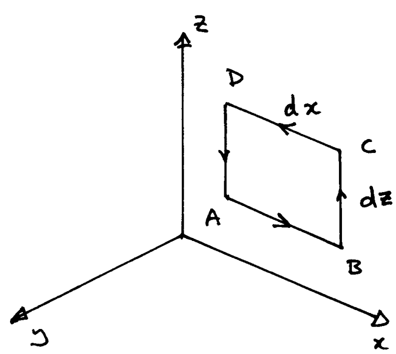


7.12 Differential form of Faraday's law

Consider

$$\oint \underline{E} \bullet \mathrm{d}\underline{s} \tag{7.67}$$

for the path ABCDA shown.



The electric field at A has components E_x, E_y, E_z. Since the path lies in the xz plane, there will be no contribution to $\underline{E} \bullet \mathrm{d}\underline{s}$ from E_y (since E_y is $\perp$ x, z). Along AB and CD, only E_x will contribute. Along BC and DA only E_z will contribute. The field E_x at the four corners is

A	E_x
B	$E_x + \left(\frac{\partial E_x}{\partial x}\right) \mathrm{d}x$
C	$E_x + \left(\frac{\partial E_x}{\partial x}\right) \mathrm{d}x + \left(\frac{\partial E_x}{\partial z}\right) \mathrm{d}z$
D	$E_x + \left(\frac{\partial E_x}{\partial z}\right) \mathrm{d}z$

From this we can see that along CD, E_x exceeds the value along AB by $\left(\frac{\partial E_x}{\partial z}\right) \mathrm{d}z$ at all points, so the net contribution to $\oint \underline{E} \bullet \mathrm{d}\underline{s}$ from AB and CD is

$$-\left(\frac{\partial E_x}{\partial z}\right) \mathrm{d}z.\ \mathrm{d}x$$

where the minus sign arises because the path is in the negative x-direction along CD.

- Similarly, the contribution from BC and DA will be

$$\left(\frac{\partial E_z}{\partial x}\right) \mathrm{d}x.\ \mathrm{d}z$$

So that overall we have

$$\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)\ \mathrm{d}x\mathrm{d}z$$

- This is the y-component of curl $\underline{E} = \underline{\nabla} \times \underline{E}$.

$$\underline{\nabla} \times \underline{E} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} \tag{7.68}$$

The magnitude of the magnetic flux through the loop ABCD is

$$B_y\ \mathrm{d}x\mathrm{d}z$$

So Faraday's law gives for y-components

$$(\underline{\nabla} \times \underline{E})_y = -\frac{\partial B_y}{\partial t}$$

and similarly for the other components, to get

$$\boxed{\underline{\nabla} \times \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad (7.69)}$$

which is Faraday's law in differential form. Note that we can change from a total differential with respect to time to a partial differential since the change in flux arises only through a change in $\underline{B}$.

7.13 Differential form of the Ampère-Maxwell law

By following the same procedure outline above, it is possible to recast the Ampère-Maxwell law

$$\oint \underline{B} \cdot \mathrm{d}\underline{s} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \int \underline{\mathbf{J}} \cdot \mathrm{d}\underline{a} + \mu_0 \epsilon_0 \frac{d}{dt} \int \underline{E} \cdot \mathrm{d}\underline{a} \tag{7.70}$$

in differential form

$$\boxed{\underline{\nabla} \times \underline{B} = \mu_0 \underline{\mathbf{J}} + \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t} \qquad (7.71)}$$