3 Conductors

3.1 Electric field and electric potential in the cavity of a conductor

Conductors (eg metals) are materials in which charges move freely, whereas insulators (eg glass, rubber, wood) are materials in which charges do not move freely.

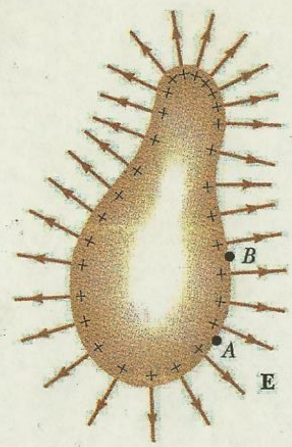


Figure 27 An arbitrarily shaped conductor carrying a positive charge. When the conductor is in electrostatic equilibrium, all of the charge resides at the surface, **E** = 0 inside the conductor, and the direction of **E** just outside the conductor is perpendicular to the surface. The electric potential is constant inside the conductor and is equal to the potential at the surface. Note from the spacing of the plus signs that the surface charge density is nonuniform.

We can deposit a net charge on a conductor. If left isolated, the charges will distribute themselves to achieve electrostatic equilibrium ie so that they become stationary.

A conductor in electrostatic equilibrium has the following properties:

1. The electric field is zero everywhere inside a conductor (otherwise the charges would not be stationary, since $\underline{F} = q\underline{E}$).

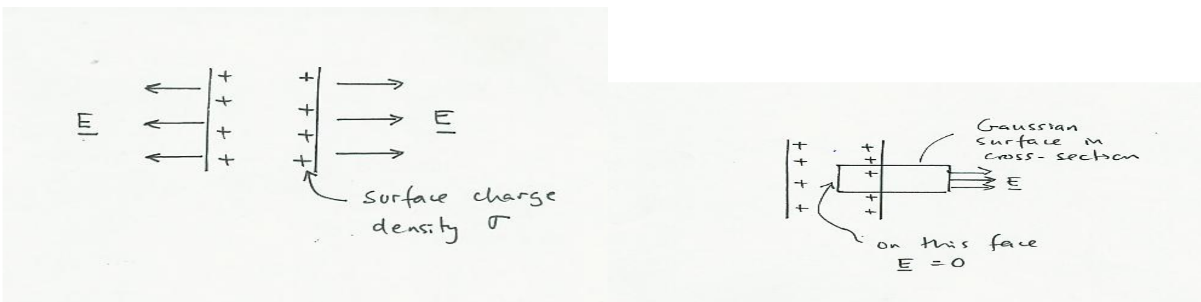
2. If an isolated conductor carries a charge, the charge resides on its surface. Gauss' law tells us that if there is no field within the conductor there can be no charge contained within it.

3. The electric field just outside a charged conductor is perpendicular to the surface of the conductor (the tangential component of $\underline{E}$ at the surface must be zero, otherwise the charges would move).

It is straighforward to calculate the electric field just outside a conductor as a function of the surface charge density σ. Indeed by using Gauss' law, and considering that: (a) the electric field is zero inside the conductor, (b) the tangential component of E is zero just outside the conductor, we find that the component E_n of the electric field normal to the surface is

$$E_n = \frac{\sigma}{\epsilon_0} \ . \tag{3.1}$$

3.2 Field from an infinite conducting plate



Consider a plate of finite width carrying an excess positive charge. In isolation, the charge will migrate to both surfaces.

To evaluate the electric field, we use Gauss' law. Take a cylindrical surface, as before, but with one face inside the conductor.

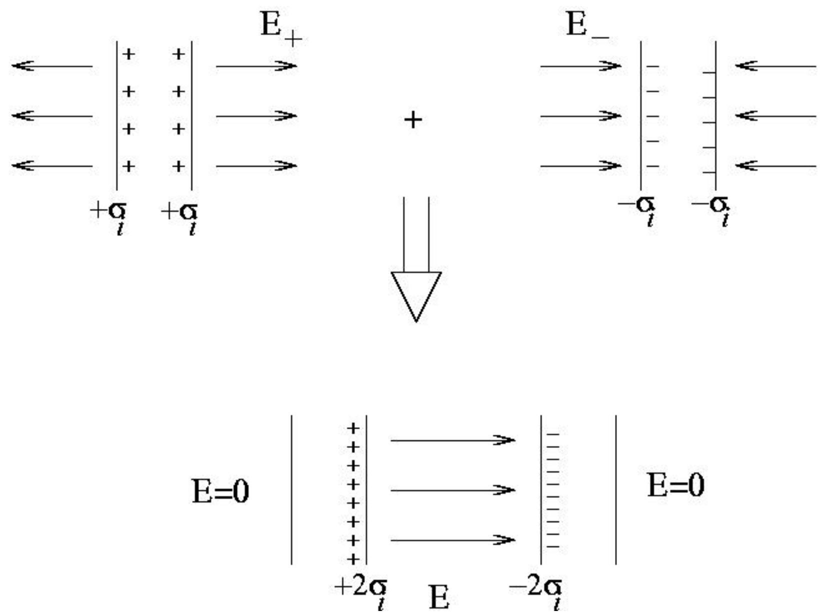
The LHS of Gauss' law is now EA and the RHS is $\frac{\sigma A}{\epsilon_0}$, so

$$\boxed{E = \frac{\sigma}{\epsilon_0}, \qquad (3.2)}$$

which is double that for an insulator with the same surface charge density σ.

3.3 Field of two, oppositely charged conducting plates

If we have two infinite plates with equal and opposite excess charge, then bring them together:



where between the plates

$$E = \frac{2\ \sigma_i}{\epsilon_0} \tag{3.3}$$

and

$$E = 0 \tag{3.4}$$

outside the pair of plates.

If the positively charged plate is in isolation there is no field within it. The nearby negatively charged plate causes an electric field which drives the positive charges to the right in the diagram. Similarly the negative charges are driven to the left. All the charge resides on the inner surfaces of the conductors with twice the density of the original distributions.

3.4 Fields outside charged conductors: method of images

It is in general quite difficult to calculate the electric field and potential outside charged conductors. Indeed this requires to solve the equation

$$\underline{\nabla} \cdot \underline{E} = \frac{\rho}{\epsilon_0} \tag{3.5}$$

which, substituting $\underline{E} = -\underline{\nabla} V$, becomes the Poisson equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \,, \tag{3.6}$$

where we introduced the operator

$$\nabla^2 = \underline{\nabla} \cdot \underline{\nabla} \,. \tag{3.7}$$

Poisson equation is then completed by the boundary conditions: the potential is constant over the conductor. To solve Eq. (3.6) is in general a difficult task. In some cases the "method of electrical images" allows us to easily solve the problem. This method consists in replacing a conductor by a point charge placed so that the surface previously corresponding to the conductor surface is still an equipotential surface.

Example: Find the electric potential for a point charge q placed at a distance a from an infinite conducting plate at zero potential.

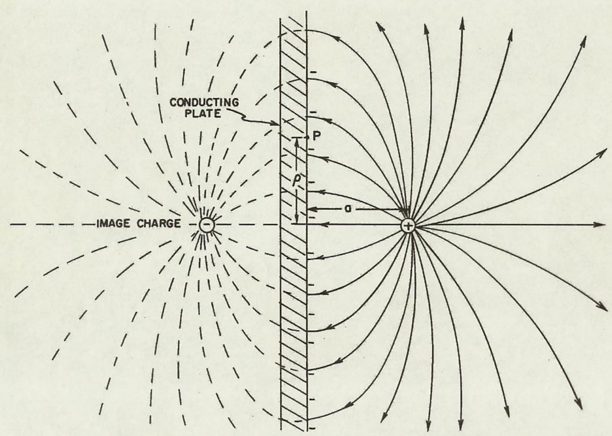


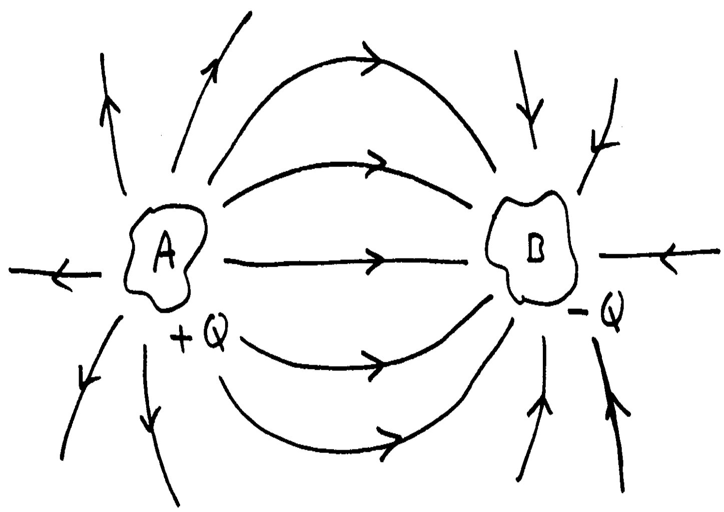
Fig. 28 The field of a charge near a plane conducting surface, found by the method of images.

Solution: It is easy to see that $V = 0$ behind the conductor: this is a solution of Poisson equation and satisfy the boundary conditions. To find the potential on the other side of the conductor, we use the method of images. We replace the conductor by a charge $-q$ at a distance a behind the plane, so that the surface previously occupied by the conductor is still an equipotential plane. The potential is then

$$V(P) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{r} - \frac{1}{\sqrt{r^2 + 4a^2 + 4ar\cos\theta}} \right] \,. \tag{3.8}$$

3.5 Vacuum capacitor: definition of capacitance

Any combination of two conductors carrying charges of equal magnitude but opposite sign is called a capacitor.



Regardless of the shape of the conductors, they are referred to as 'plates'. There is a field $\underline{E}$ between the plates A and B, and a potential difference $\Delta V = |V_{\mathrm{A}} - V_{\mathrm{B}}|$. Note that ΔV is always expressed as a positive quantity.

The priciple of superposition implies that "the amount of charge Q on a capacitor" (ie on either plate, not the net charge on both plates, which equals zero) is proportional to ΔV, which we write as

$$\boxed{Q = C\Delta V \qquad (3.9)}$$

where the capacitance

$$C = \frac{Q}{\Delta V} \tag{3.10}$$

is a constant that depends on the shape and separation of the conductors. C is always positive. It is a measure of a capacitor's ability to store charge and electric potential energy. The SI unit of capacitance is the farad (F), and one farad = one coulomb per volt which is a very large unit. In practice values range from pico (10^{-12}) to nano (10^{-9}) to micro (10^{-6}). One microfarad = 1μF = 10^{-6}F. Note: As the unit of potential difference ΔV is the volt, ΔV is often called the 'voltage' eg 'the voltage between the plates'.

3.6 Parallel-plate capacitors

We have seen in Sec. 3.3 that for two infinite plates, Gauss' law gives

$$E = \frac{\sigma}{\epsilon_0} \tag{3.11}$$

where σ is the surface charge density on either plate.

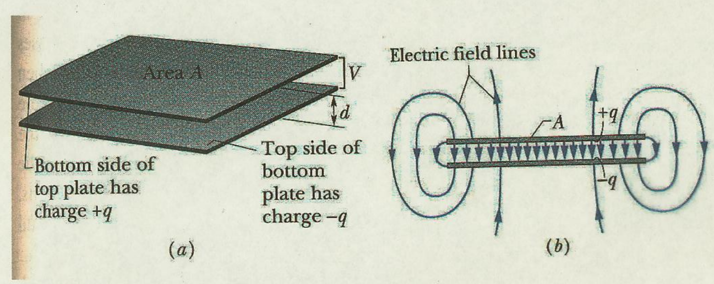


Fig. 33 (a) A parallel-plate capacitor, made up of two plates of area A separated by a distance d. The plates have equal and opposite charges of magnitude q on their facing surfaces. (b) As the field lines show, the electric field due to the charged plates is uniform in the central region between the plates. The field is not uniform at the edges of the plates, as indicated by the 'fringing' of the field lines there.

If the plates are of area A and carry a charge $\pm q$, then $\sigma = \frac{q}{A}$, and the potential difference

$$\Delta V = Ed = \frac{qd}{\epsilon_0 A} \tag{3.12}$$

where d is the separation of the plates. Note that equation 3.12 corresponds

$$\Delta V = |V(x=0) - V(x=d)| = \int_0^d E\mathrm{d}x = E[x]_0^d = Ed \tag{3.13}$$

since $\underline{E}$ is uniform and in the x-direction.

So, ignoring edge effects (non-uniform $\underline{E}$),

$$\boxed{C = \frac{q}{\Delta V} \simeq \frac{\epsilon_0 A}{d} \qquad (3.14)}$$

which is bigger for larger plates and smaller separations.

3.7 Spherical capacitors

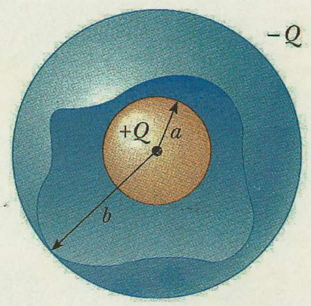


Figure 35 A spherical capacitor consists of an inner sphere of radius a surrounded by a concentric spherical shell of radius b. The electric field between the spheres is directed radially outward when the inner sphere is positively charged.

Imagine two concentric spherical shells. Note that for $r > b$, the net enclosed charge is zero so Gauss' law gives $\underline{E} = 0$, $V =$ constant. Between the shells, $a \leq r \leq b$, the potential is the same as that of a point charge Q, so that at $r = b$

$$V_b = \frac{1}{4\pi\epsilon_0}\frac{Q}{b} \tag{3.15}$$

and the potential of the inner shell (radius a) is

$$V_a = \frac{1}{4\pi\epsilon_0}\frac{Q}{a} \tag{3.16}$$

Therefore

$$\Delta V = |V_a - V_b| = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right) = \frac{Q}{4\pi\epsilon_0}\frac{b-a}{ab} \tag{3.17}$$

therefore

$$\boxed{C = \frac{Q}{\Delta V} = 4\pi\epsilon_0\frac{ab}{b-a} \qquad (3.18)}$$

We can use 3.18 to derive the capacitance of an isolated spherical conductor: as $b \to \infty$, $\frac{b}{b-a} \to 1$, therefore

$$\boxed{C_{\text{single sphere}} = 4\pi\epsilon_0 a \qquad (3.19)}$$

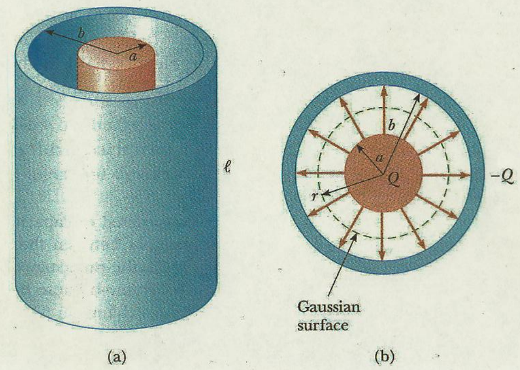


Figure 36 (a) A cylindrical capacitor consists of a solid cylindrical conductor of radius a and length ℓ surrounded by a coaxial cylindrical shell of radius b. (b) End view. The dashed line represents the end of the cylindrical gaussian surface of radius r and length ℓ.

3.8 Cylindrical capacitors

A solid conductor of radius a, a cylindrical shell of radius b and length l. If $l \gg b$ we can neglect end effects and use the result for an infinite line of charge with density λ, that

$$\underline{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}\,\underline{\hat{r}} \qquad a \leq r \leq b \tag{3.20}$$

where we put $\lambda = Q/l$.

Note the physical assumptions and arguments here -

- Neglecting end effects means that the field is radially outward and perpendicular to the axis.
- The charge on the outer cylinder does not contribute to the field inside it (Gauss' law).
- Outside the central conductor, $r \geq a$, the field is that due to a line of charge.

The potential difference between a and b is

$$\Delta V = |V_a - V_b| = \int_a^b \underline{E}\cdot \mathrm{d}\underline{r} = \frac{Q}{2\pi\epsilon_0 l}\int_a^b \frac{\mathrm{d}r}{r} = \frac{Q}{2\pi\epsilon_0 l}\ln\left(\frac{b}{a}\right) \tag{3.21}$$

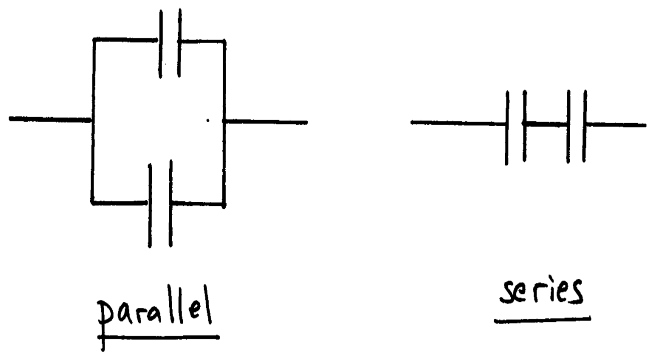
therefore

$$C = \frac{Q}{\Delta V} = \frac{2\pi\epsilon_0 l}{\ln(\frac{b}{a})} \tag{3.22}$$

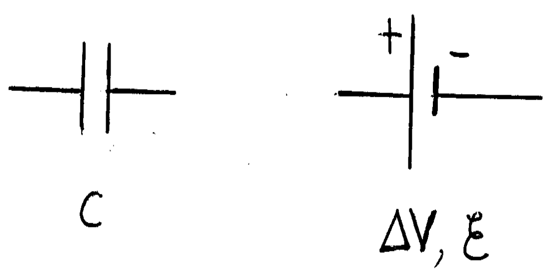
Note that the cylindrical geometry is that of a coaxial cable as used in electricity and signal transmission.

3.9 Combinations of capacitors

In electric circuits we often use two or more capacitors either arranged 'in parallel' or 'in series', represented as



where the capacitor symbol

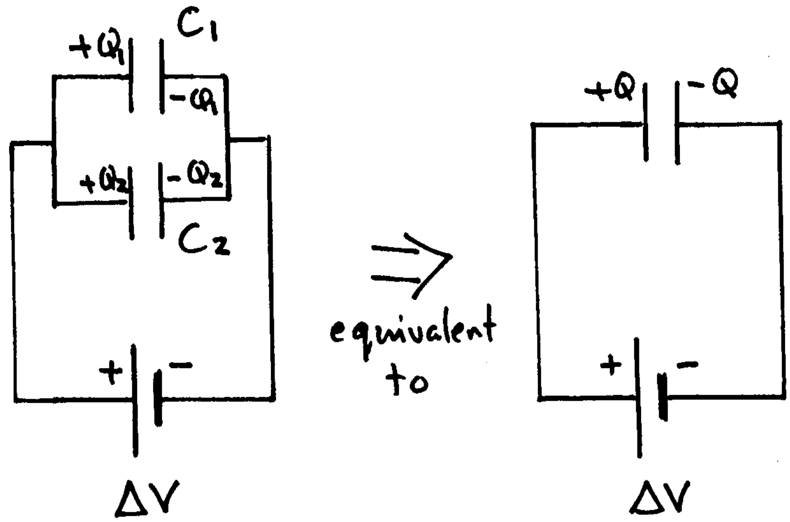


reflects the common parallel-plate design. Note the difference between the capacitor and battery symbols.

Parallel combination

What single capacitor has the same capacitance as two others wired in parallel?

The wires are conductors - they connect things with the same potential. Thus the potential difference across the battery terminals ΔV is the same as across the plates of C_1 and C_2.



$$\Delta V = \frac{Q_1}{C_1} = \frac{Q_2}{C_2} \tag{3.23}$$

In the equivalent circuit we want to store the same total charge

$$\Delta V = \frac{Q}{C_{\text{parallel}}}, \qquad Q = Q_1 + Q_2 \tag{3.24}$$

Combining these two equations

$$C_{\text{parallel}} = \frac{Q}{\Delta V} = \frac{Q_1 + Q_2}{\Delta V} = \frac{C_1 \Delta V + C_2 \Delta V}{\Delta V} \tag{3.25}$$

therefore

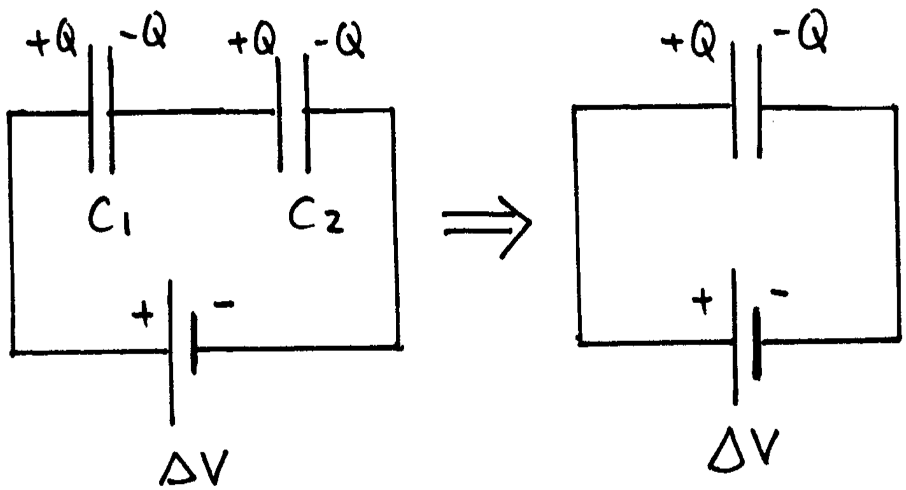
$$\boxed{C_{\text{parallel}} = C_1 + C_2 \qquad (3.26)}$$

The capacitance increases.

<u>Series combination</u>

What single capacitor has the same capacitance as two wired in series?

In the diagram, the right hand plate of C_2 and the left hand plate of C_1 are connected to the battery terminals. The battery drives electrons away from the negative terminal



so that they accumulate on the right hand plate of C_2, giving the charge $-Q$. An equal and opposite charge must accumulate on the left hand plate of C_1.

Now consider the right hand plate of C_1 and the left hand plate of C_2. They are connected but isolated from the rest of the circuit. If no battery is present the net charge on these two plates is zero and it must remain so once a battery is connected. So they must carry equal and opposite charges also. The positive charge on the left hand plate of C_1 creates an electric field that attracts negative charge to the right hand plate of C_1. This process will continue until there is electrostatic equilibrium which occurs when the charges on the plates are equal in magnitude (but of opposite sign).

The final configuration has charge of the same magnitude on all four plates as shown. The potential difference across each capacitor may be different as shown.

The total potential difference is

$$\Delta V = \Delta V_1 + \Delta V_2 = \frac{Q}{C_1} + \frac{Q}{C_2} \tag{3.27}$$

will be the same in the equivalent circuit

$$\Delta V = \frac{Q}{C_{\text{series}}} \tag{3.28}$$

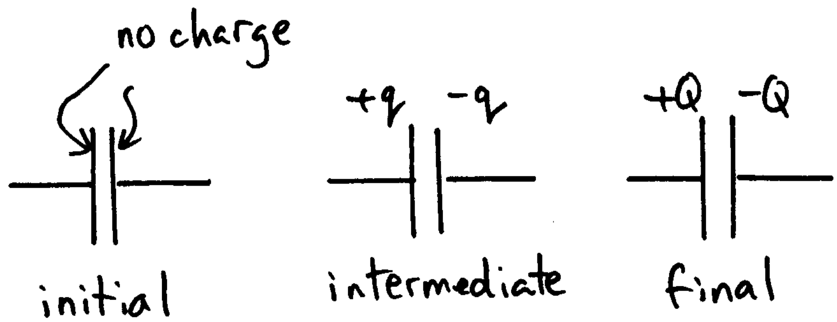
therefore

$$\boxed{\frac{1}{C_{\text{series}}} = \frac{1}{C_1} + \frac{1}{C_2} \qquad (3.29)}$$

the capacitance <u>decreases</u>

3.10 Energy stored in a charged capacitor?

How much work is done to charge a capacitor?



Imagine that charge is passed from one plate to the other, <u>against</u> the electric field between the plates. The work done to transfer an element of charge $\mathrm{d}q$ when the plates carry a charge q and the potential difference between the plates is ΔV is

$$\mathrm{d}W = \Delta V \mathrm{d}q \qquad (3.30)$$

where $\Delta V = q/C$. Remember that ΔV is the work required to move a <u>unit</u> charge.

The total work done is

$$W = \int_0^Q \frac{q}{C}\mathrm{d}q = \frac{1}{2}\,\frac{Q^2}{C} \qquad (3.31)$$

which equals the total electric potential energy U stored in the capacitor:

$$\boxed{U = \frac{1}{2}\,\frac{Q^2}{C} \qquad (3.32)}$$

or, since $C = Q/\Delta V$

$$U = \frac{1}{2}\,Q\,\Delta V \qquad (3.33)$$

or

$$U = \frac{1}{2}\ C\ (\Delta V)^2 \tag{3.34}$$

These results apply to any capacitor, whatever its geometry. Note that sometimes ΔV is written just as V, so $U = \frac{1}{2}QV$ etc. Devices that use capacitors to store and rapidly deliver electrical energy to a conductor include

1. the medical defibrillator, the conductor in this case being the human heart.

2. the camera flash unit - the conductor is the flash bulb.

3.11 Energy density

The potential energy stored in a capacitor can be seen as being stored in the electric field. eg a parallel plate capacitor, plate area A, plate separation d. Neglecting edge effects

$$E = \frac{\Delta V}{d} \tag{3.35}$$

The volume of the capacitor is $A\ d$, so the energy density, u, (energy stored per unit volume) is

$$u = \frac{U}{A\ d} \tag{3.36}$$

Now

$$U = \frac{1}{2}\ C\ (\Delta V)^2 = \frac{1}{2}\ d^2\ C\ E^2 \tag{3.37}$$

We saw that (equation 3.14) we saw that $C = \frac{\epsilon_0\ A}{d}$ therefore

$$u = \frac{U}{A\ d} = \frac{\frac{1}{2}\ d^2 \frac{\epsilon_0\ A}{d}\ E^2}{a\ d} = \frac{1}{2}\ \epsilon_0\ E^2 \tag{3.38}$$

This result applies generally to the energy density of a charged capacitor.