A family of efficient six-regular circulants representable as a Kronecker product

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Abstract: Broere and Hattingh proved that the Kronecker product of two circulants whose orders are co-prime is a circulant itself. This paper builds on this result to construct a family of efficient three-colorable, six-regular circulants representable as the Kronecker product of a Möbius ladder and an odd cycle. The order of each graph is equal to $4d^2 - 2d - 2$ where d denotes the diameter and $d \equiv 3, 5 \pmod 6$. Additional results include (a) distance-wise vertex distribution of the circulant leading to its average distance that is about two-thirds of the diameter, (b) routing via shortest paths, and (c) an embedding of the circulant on a torus with a half twist. In terms of the order-diameter ratio and odd girth, the circulants in this paper surpass the well-known triple-loop networks having diameter d and order $3d^2 + 3d + 1$.

Key words: Six-regular circulants; Kronecker product; Möbius ladder; twisted torus; network topology; routing; embedding; graphs and networks.

1 Introduction

Circulant graphs constitute a subfamily of Cayley graphs [17]. They possess attractive properties such as high symmetry, high connectivity and scalability, which lend them to an application as a *network topology* in areas like parallel computers, distributed systems and VLSI [3, 4, 5, 19, 25, 32, 29, 34].

Over the years, the four-regular circulants have received a lot of attention leading to sharp results relating to their isomorphism, connectivity, routing, and several other issues. See the surveys [3, 19, 27] and the references therein. On the other hand, higher-degree circulants have not been studied at length. This is probably because the associated problems in the latter case are relatively challenging [18].

This paper presents a family of *six-regular circulants*, each representable as the *Kronecker product* of the *Möbius ladder* of order p and the *odd cycle* C_{p+3}, where $p \equiv 4, 8 \pmod{12}$. The order of each graph turns out to be $4d^2 - 2d - 2$, where d denotes its diameter and $d \equiv 3, 5 \pmod 6$.

Prominent circulant graphs of degree six include the triple-loop networks [35], hexagonal meshes [9] and Eisenstein-Jacobi networks [26]. It turns out that $\mathcal{C}_{3d^2+3d+1}(1, 3d+1, 3d+2)$ is a representative in these cases [31], where d is its diameter. See Table 1 for a comparison, where *order-diameter ratio* refers to order of the graph relative to its diameter. (The definitions and proofs appear later.)

Each graph admits an *embedding* (with edge crossings) on a *torus* with a half twist. Additionally, (a) the graph admits an efficient routing via shortest paths, (b) its odd girth is equal to $2d+1$, where d denotes its diameter, (c) its average distance is about two-thirds of the diameter, (d) its chromatic number is equal to three, (e) its vertex connectivity is equal to its degree that is six, and (f) it admits a Hamiltonian decomposition. (High odd girth, low average distance, low chromatic number, high connectivity and Hamiltonian

Table 1: Comparison of $\mathcal{C}_{4d^2-2d-2}(1, s, t)$ and $\mathcal{C}_{3d^2+3d+1}(1, 3d+1, 3d+2)$

	$\mathcal{C}_{4d^2-2d-2}(1, s, t)$		$\mathcal{C}_{3d^2+3d+1}(1, 3d+1, 3d+2)$
order-diameter ratio	$4d$ (approx.)	⌣̈	$3d$ (approx.)
odd girth	$2d+1$	⌣	3
chromatic number	3		3
connectivity	6		6

$$s = \tfrac{1}{3}d(2d+1) - 1 \text{ and } t = 2s+1 \text{ if } d \equiv 3 \pmod 6$$
$$s = \tfrac{1}{3}(d-2)(2d+1) + 1 \text{ and } t = 2s-1 \text{ if } d \equiv 5 \pmod 6$$

decomposition are a big plus in a network.)

1.1 Definitions and preliminaries

When we speak of a graph, we mean a finite, simple, undirected and connected graph. Let G be a graph, and let $d_G(u,v)$ denote the shortest distance between vertices u and v in G. Further, let diam(G) represent its diameter, i.e., $\max\{d_G(u,v)\colon u,v \in V(G)\}$. A distance-preserving subgraph of a graph is called an *isometric subgraph* [16]. We employ *vertex* and *node* as synonyms, and write "G is isomorphic to H" as $G \cong H$.

Say that a vertex v in G is at *level i* relative to a fixed vertex u if $d_G(u,v) = i$. A *level diagram* of G relative to u consists of a layout of the graph in which vertices at a distance of i from u appear on a "line at a height" of i above u, for $0 \leq i \leq$ diam(G). Vertices at a distance of diam(G) from u are called *diametrical* relative to u. If G is known to be vertex-transitive, a property held by a circulant [17], then the form of its level diagram is independent of the choice of the source vertex. A related term is *distance-wise vertex count* that refers to the number of vertices at each of the i-th level, $0 \leq i \leq$ diam(G).

The *average distance* of a graph $G = (V,E)$ relative to a fixed vertex u is given by $\left(\frac{1}{|V|}\sum_{i \geq 0} i n_i\right)$, where n_i represents the number of vertices at a distance of i from u in G. Vertex transitivity of a circulant ensures that its average distance is computable relative to any vertex, and the sum in the previous expression runs from $i = 0$ to $i =$ diam(G).

The *Kronecker product* (also known as the tensor product, direct product, and graph conjunction) $G \times H$ of graphs $G = (U,D)$ and $H = (W,F)$ is defined as follows: $V(G \times H) = U \times W$, and $E(G \times H) = \{\{(a,x),(b,y)\} \mid \{a,b\} \in D \text{ and } \{x,y\} \in F\}$. It is one of the most important products, with numerous applications [16].

Let C_n denote the cycle on the vertex set $\{0,\dots,n-1\}$, $n \geq 3$, where adjacencies $\{i,i+1\}$ exist in the natural way. A spanning cycle in a graph (if one exists) is called a *Hamiltonian cycle*. Further, a graph is said to admit a *Hamiltonian decomposition* if its edge set can be partitioned into Hamiltonian cycles. The length of a shortest (induced) odd cycle in a nonbipartite graph G is called its *odd girth*, denoted by og(G). Let $\chi(G)$ denote the *chromatic number* of G, and let $\kappa(G)$ denote its *vertex connectivity*. For undefined terms, see Hammack *et al.* [16].

Proposition 1.1 *[16, 7]*

(1) $G \times H$ is connected iff G and H are both connected and at least one of them is nonbipartite.

(2) The degree of a vertex (u,v) in $G \times H$ is equal to the product of the degrees of u and v in the respective graphs.

(3) $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$.

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(4) If G and H are both nonbipartite, then $og(G \times H) = max\{og(G), og(H)\}$.

(5) If G and H are both vertex-transitive, then so is $G \times H$. ∎

Let $n, s_1, \ldots, s_k$ be such that $n \geq 3$, and $1 \leq s_1 < s_2 < \ldots < s_k \leq \lfloor n/2 \rfloor$. The circulant $\mathcal{C}_n(s_1, \ldots, s_k)$ is a graph on the vertex set $\{0, \ldots, n-1\}$, where each vertex i is adjacent to each of $(i \pm s_1) \bmod n, \ldots, (i \pm s_k) \bmod n$. The parameters $s_1, \ldots, s_k$ are called the *step sizes* or *jumps*. If n is even and $s_k = n/2$, then the graph is $(2k-1)$-regular, otherwise it is $2k$-regular. It is known that (i) $\mathcal{C}_n(s_1, \ldots, s_k)$ is connected iff $\gcd(n, s_1, \ldots, s_k) = 1$, and (ii) $\mathcal{C}_n(s_1, \ldots, s_k)$ is bipartite iff n is even, and each of $s_1, \ldots, s_k$ is odd. Here is the baseline of the present study.

Proposition 1.2 *[7] If G and H are circulants such that $|V(G)|$ and $|V(H)|$ are co-prime, then $G \times H$ is a circulant itself.* ∎

1.2 Related work

Yebra *et al.* [35] were probably the first to study six-regular circulants. Among other things, they proved that the maximum order of $\mathcal{C}_n(a, b, a+b)$ with diameter d is equal to $n_d := 3d^2 + 3d + 1$. They further showed that $\mathcal{C}_{n_d}(1, 3d+1, 3d+2)$ achieves the preceding upper bound, and that this graph is amenable to a hexagonal tessellation. Interestingly enough, this circulant arose in projects such as HARTS (Hexagonal Architecture for Real-Time System) [9], Mayfly [10] and FAIM-1 [11]. Thomson and Zhou recently showed that $\mathcal{C}_{n_d}(1, 3d+1, 3d+2)$ admits optimal routing and gossiping [30], and it is a special instance of an interesting family of first-kind Frobenius circulants of degree six [31]. (See the references therein for relevant definitions.)

In another development, Martínez *et al.* [26] came up with a family of graphs called EJ networks based on the Eisenstein-Jacobi integers. It turns out that the six-regular first-kind Frobenius graph [31] is exactly the EJ network $EJ_{a+b\rho}$ with $\gcd(a, b) = 1$ and order congruent to 1 modulo 6, where $\rho = \frac{1}{2}(1 + \sqrt{-3})$. For other studies, see Decayeux and Semé [13] and Garcia *et al.* [14], and for theoretical upper bounds on the order of a six-regular circulant relative to its diameter, see Aguiló-Gost [2] and the surveys [3, 19, 27].

1.3 Möbius ladder

For p even and $p \geq 4$, the *Möbius ladder M_p* is a three-regular graph on p vertices obtainable from the cycle C_p by introducing edges that connect "opposite" pairs of vertices in the cycle. Fig. 1(a) illustrates this definition in respect of M_{16}, while Fig. 1(b) presents another view of the same graph.

Möbius ladders play an important role in the study of graph minors [33], and integer programming approaches to solve problems in set packing and linear ordering [6].
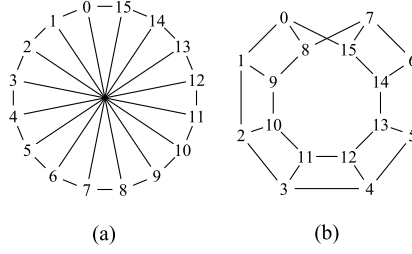
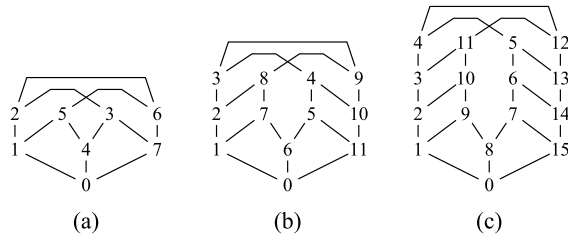Figure 1: Two drawings of the Möbius ladder M_{16}

Figure 2: Level diagrams of (a) M_8, (b) M_{12} and (c) M_{16}

Proposition 1.3 *[1, 15, 24] Let $p \equiv 0 \pmod{4}$.*

(1) $M_p \cong \mathcal{C}_p(1, \frac{1}{2}p)$.

(2) $\chi(M_p) = 3$.

(3) $og(M_p) = \frac{1}{2}p + 1$.

(4) $diam(M_p) = \frac{1}{4}p$.

(5) $\kappa(M_p) = 3$. ∎

Proposition 1.4 *[24] For $p \equiv 0 \pmod{4}$, the distance-wise vertex partition of M_p is as follows:*

(1) Level $0 : \{0\}$

(2) Level $1 : \{1, \frac{1}{2}p, p - 1\}$

(3) Level $i : \{i, \frac{1}{2}p - i + 1, \frac{1}{2}p + i - 1, p - i\}$, where $2 \leq i \leq \frac{1}{4}p$. ∎

See Fig. 2 for level diagrams of M_8, M_{12} and M_{16}, each relative to vertex 0. The (four) nodes at the diametrical level of M_p induce a path, viz., $(\frac{1}{4}p + 1) - (\frac{1}{4}p) - (\frac{3}{4}p) - (\frac{3}{4}p - 1)$ that is of length three. On the other hand, nodes at each of the lower levels are mutually nonadjacent.

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Definition 1 *The nodes $0, 1, \ldots, \frac{1}{2}p$ in M_p induce an odd cycle, which we refer to as the canonical cycle $C_{\frac{1}{2}p+1}$, where $p \equiv 0 \ (mod\ 4)$.* ∎

Lemma 1.5 *The canonical cycle is an isometric cycle whose diameter is equal to that of M_p itself.*

Proof. The length of the canonical cycle is equal to the odd girth of the graph. Further, a shortest odd cycle in a graph is known to be an isometric subgraph [16] (Prop. 3.3, p. 29). Finally, it is clear that the diameter of $C_{\frac{1}{2}p+1}$ is equal to $\frac{1}{4}p$ that coincides with the diameter of M_p itself. ∎

Lemma 1.6 *For every walk between two nodes u and v in M_p, where u and v lie on the canonical cycle, there exists a walk of the same length between u and v in which all nodes belong to the canonical cycle itself.*

Proof. Let u and v be as stated, where $0 \leq u \leq v \leq \frac{1}{2}p$. It is clear that there are two simple paths, say P and Q, between u and v along the canonical cycle. Their lengths are $|u - v|$ and $\frac{1}{2}p + 1 - |u - v|$, respectively, which themselves are of distinct parities. Assume without loss of generality that $|P| = |u - v| < |Q| = \frac{1}{2}p + 1 - |u - v|$.

Consider a walk, say W, from u to v in which all intermidiate nodes are from $\{\frac{1}{2}p + 1, \ldots, p - 1\}$. The canonical cycle being an isometric cycle, every shortest path between u and v must be of length $|P|$. If $|W|$ and $|P|$ are of the same parity, then $|W| \geq |P|$ and $|W| - |P|$ is even. Here is a desired walk between u and v: Follow the path P, and retrace one of its edges $|W| - |P|$ times.

On the other hand, if $|W|$ and $|Q|$ are of the same parity, then $|W| \geq |Q|$. This is because every odd cycle in M_p is of length at least $\frac{1}{2}p + 1$, hence a path between u and v whose length is of a parity different from that of $|P|$ must be of length at least $|Q|$. In this case, too, there exists a walk between u and v that is of the same length as W and that is obtainable from Q by retracing one of its edges $|W| - |Q|$ times. ∎

1.4 Distances in the Kronecker product

Unlike the (shortest) distance between two vertices in other products, the distance in the Kronecker product is based on a shortest even walk and a shortest odd walk (and the respective even distance and odd distance) between two vertices in the factor graphs [22]. To that end, let $d_G^e(a, b)$ and $d_G^o(a, b)$ denote the shortest even distance and the shortest odd distance, respectively, between vertices a and b in a graph G. (If G is nonbipartite, then the even distance and the odd distance between two vertices, not necessarily distinct, are well-defined and finite.)

Proposition 1.7 *[22] If G and H are both nonbipartite graphs, then $d_{G \times H}((a, x), (b, y)) = min\{max\{d_G^e(a, b), d_H^e(x, y)\}, max\{d_G^o(a, b), d_H^o(x, y)\}\}$.* ∎

Proposition 1.8 *[22] If m and n are both odd and $m \geq n$, then*

$$diam(C_m \times C_n) = \begin{cases} n - 1 & \text{if } m = n \\ max\{\frac{1}{2}(m - 1), n\} & \text{if } m > n. \end{cases} ∎$$

Structure of the paper

Sec. 2 presents the main result, viz., $M_p \times C_{p+3}$ is a low-diameter six-regular circulant, while Sec. 3 computes the average distance of the graph. Whereas Sec. 4 presents a routing scheme via shortest paths, Sec. 5 determines the precise jump sequence associated with the circulant, and Sec. 6 presents an embedding (with edge crossings) of the circulant on a torus with a half twist. The paper ends with certain concluding remarks in Sec. 7.

2 Main result

This section shows that $diam(M_p \times C_n)$ reaches its minimum relative to its order iff $n = p+3$, where $p \equiv 0 \pmod 4$. That, in turn, leads to the low-diameter six-regular circulant $M_p \times C_{p+3}$, where $p \equiv 4, 8 \pmod{12}$.

Lemma 2.1 *For $p \equiv 0 \pmod 4$ and n odd, $C_{\frac{1}{2}p+1} \times C_n$ is an isometric subgraph of $M_p \times C_n$, where $C_{\frac{1}{2}p+1}$ is the canonical cycle $0 - 1 - \ldots - \frac{1}{2}p - 0$ of M_p.*

Proof. The distance between two nodes cannot go down in a subgraph. In that light, let $(u, i), (v, j) \in V(C_{\frac{1}{2}p+1} \times C_n)$, and let $P = (w_1, x_1) - \ldots - (w_r, x_r)$ be a shortest path between (u, i) and (v, j) in $M_p \times C_n$, where $w_1 = u$, $w_r = v$, and $x_1 = i$, $x_r = j$. In the process, (a) $w_1, \ldots, w_r$ is a sequence of nodes corresponding to a walk between u and v in M_p, and (b) $x_1, \ldots, x_r$ is a sequence of nodes corresponding to a walk between i and j in C_n.

By Lemma 1.6, there exists a sequence $y_1, \ldots, y_r$ that defines a walk between u and v in which each y_k belongs to the canonical cycle itself, where $y_1 = u$, $y_r = v$. That, in turn, leads to the path $(y_1, x_1), \ldots, (y_r, x_r)$ between (u, i) and (v, j) that lies entirely in $C_{\frac{1}{2}p+1} \times C_n$. The length of this path is equal to that of P, hence the distance between (u, i) and (v, j) relative to the subgraph $C_{\frac{1}{2}p+1} \times C_n$ is equal to that between the same two vertices relative to $M_p \times C_n$ itself. Isometry follows. ∎

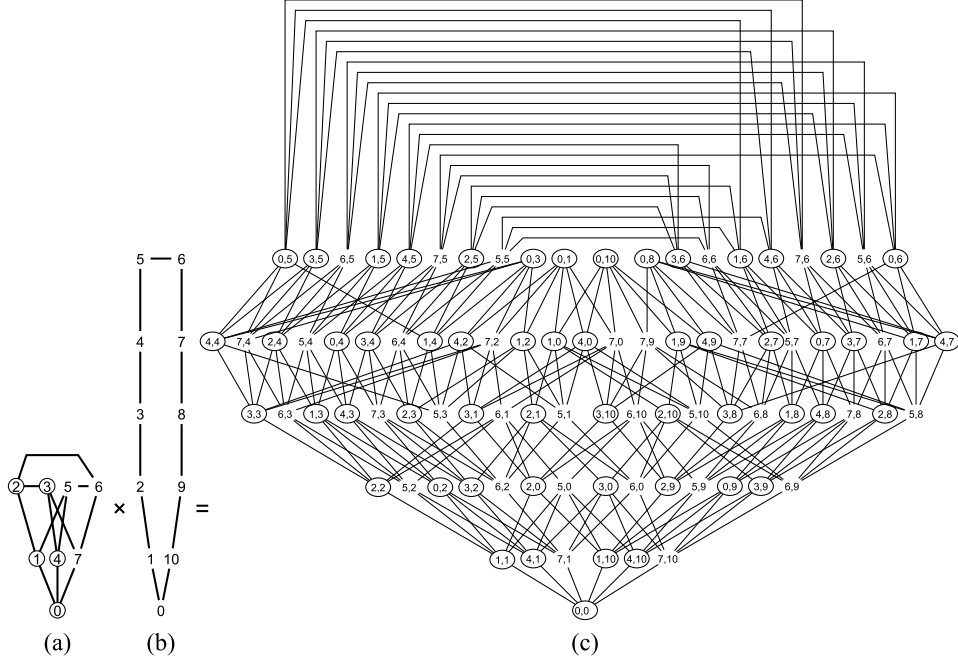**Lemma 2.2** *For $p \equiv 0 \pmod 4$ and n odd, $diam(M_p \times C_n) = diam(C_{\frac{1}{2}p+1} \times C_n)$.*

Figure 3: The level diagrams of (a) M_8, (b) C_{11} and (c) $M_8 \times C_{11}$

Proof. By Lemma 2.1, $\mathrm{diam}(M_p \times C_n) \geq \mathrm{diam}(C_{\frac{1}{2}p+1} \times C_n)$. For the reverse inequality, it suffices to show that the set of diametrical vertices of $M_p \times C_n$ relative to $(0,0)$ includes at least one node that belongs to $C_{\frac{1}{2}p+1} \times C_n$. To that end, let (v, i) be a vertex that is in $M_p \times C_n$ but not in $C_{\frac{1}{2}p+1} \times C_n$. Then $\frac{1}{2}p+1 \leq v \leq p-1$. Note that $v - \frac{1}{2}p+1 \in V(C_{\frac{1}{2}p+1})$. By Prop. 1.4(3), v and $v - \frac{1}{2}p+1$ are equidistant from 0 in M_p. (See Fig. 2.) By symmetry between v and $v - \frac{1}{2}p+1$ relative to 0 in M_p, (v, i) is a diametrical vertex of $M_p \times C_n$ iff so is $(v - \frac{1}{2}p+1, i)$. See Fig. 3 that presents the level diagrams of M_8, C_{11} and $M_8 \times C_{11}$. The nodes belonging to the canonical cycle C_5 in M_8 and those belonging to $C_5 \times C_{11}$ in $M_8 \times C_{11}$ are "circled." ∎

Corollary 2.3 *If $p \equiv 0 \ (mod \ 4)$ and n is odd, then*

$$
diam(M_p \times C_n) = \begin{cases}
\frac{1}{4}p & 3 \leq n \leq \frac{1}{4}p \\
n & \frac{1}{4}p+1 \leq n < \frac{1}{2}p+1 \\
n-1 & n = \frac{1}{2}p+1 \\
\frac{1}{2}p+1 & \frac{1}{2}p+3 \leq n \leq p+3 \\
\frac{1}{2}(n-1) & n \geq p+5.
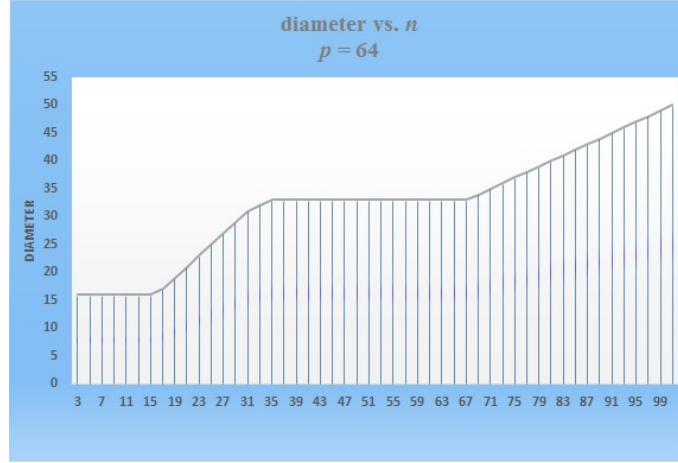\end{cases}
$$

Proof. By Prop. 1.8 and Lemma 2.2. ∎

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Figure 4: $\text{diam}(M_{64} \times C_n)$ vs. n, $3 \le n \le 101$

Table 2: Determining the minimality of $\frac{1}{n}\text{diam}(M_p \times C_n)$ for a given p

$p \equiv 0 \pmod 4$ and n odd	$\frac{1}{n}\text{diam}(M_p \times C_n)$	minimum of $\frac{1}{n}\text{diam}(M_p \times C_n)$
$3 \le n \le \frac{1}{4}p$	$\frac{p}{4n}$	1 when $n = \frac{1}{4}p$
$\frac{1}{4}p + 1 \le n < \frac{1}{2}p + 1$	1	1
$n = \frac{1}{2}p + 1$	$1 - \frac{2}{p+2}$	$1 - \frac{2}{p+2}$
$\frac{1}{2}p + 3 \le n \le p + 3$	$\frac{p+2}{2n}$	$\frac{1}{2}(1 - \frac{1}{p+3})$ when $n = p + 3$
$n \ge p + 5$	$\frac{1}{2}(1 - \frac{1}{n})$	$\frac{1}{2}(1 - \frac{1}{p+5})$ when $n = p + 5$

Fig. 4 illustrates Corollary 2.3 in respect of $M_{64} \times C_n$, where $3 \le n \le 101$.

Lemma 2.4 *For a given $p \equiv 0 \pmod 4$ and n odd, the graph $M_p \times C_n$ achieves its least diameter relative to its order iff $n = p + 3$.*

Proof. The order of $M_p \times C_n$ being equal to pn, it suffices to find an n for which the ratio of $\text{diam}(M_p \times C_n)$ to n is the minimum. To that end, see Table 2 that itself is based on Corollary 2.3.∎

The diameter relative to the order of $M_{64} \times C_n$ reaches its minimum when $n = 67$, cf. Fig. 4.

The following is the main result of this paper.

Theorem 2.5 *(1) For $p \equiv 4, 8 \pmod{12}$, $M_p \times C_{p+3}$ is a six-regular circulant, where $diam(M_p \times C_{p+3}) = \frac{1}{2}p + 1$, $og(M_p \times C_{p+3}) = p + 3$, and $\chi(M_p \times C_{p+3}) = 3$.*

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Table 3: Set S of vertices at a distance of i from 0 in M_p, $0 \leq i \leq \frac{1}{2}p + 1$

	i even	
S	if	$|S|$
$\{0\}$	$i = 0$	1
$\{i, \frac{1}{2}p - i + 1, \frac{1}{2}p + i - 1, p - i\}$	$2 \leq i \leq \frac{1}{4}p$	4
$\{i, \frac{1}{2}p - i + 1, \frac{1}{2}p + i - 1, p - i\}$	$\frac{1}{4}p + 1 \leq i \leq \frac{1}{2}p$	$\begin{cases} 4 & \text{if } i < \frac{1}{2}p \\ 3 & \text{if } i = \frac{1}{2}p \end{cases}$

	i odd	
S	if	$|S|$
$\{1, \frac{1}{2}p, p - 1\}$	$i = 1$	3
$\{i, \frac{1}{2}p - i + 1, \frac{1}{2}p + i - 1, p - i\}$	$3 \leq i \leq \frac{1}{2}p - 1$	4
$\{0\}$	$i = \frac{1}{2}p + 1$	1

(2) The order of $M_p \times C_{p+3}$ is $4d^2 - 2d - 2$, where d denotes its diameter and $d \equiv 3, 5$ (mod 6).

Proof. For (1), refer to Prop. 1.2, Prop. 1.1(1)–(4) and Lemma 2.4. For (2), diameter $d = \frac{1}{2}p + 1$, and if $p \equiv 4$ (mod 12), then $d \equiv 3$ (mod 6), and if $p \equiv 8$ (mod 12), then $d \equiv 5$ (mod 6). Further, $p(p + 3) = 4d^2 - 2d - 2$. ∎

3   Average distance of $M_p \times C_{p+3}$

We build the distance-wise node distribution of $M_p \times C_{p+3}$ relative to $(0, 0)$ leading to the average distance of the graph, where $p \equiv 4, 8$ (mod 12). To that end, the set of nodes in M_p (resp. C_{p+3}) at a distance of i from 0 appears in Table 3 (resp. Table 4). It is easy to verify the correctness of the entries.

Lemma 3.1 *The distance-wise vertex count in $M_p \times C_{p+3}$ from Level 0 to Level $\frac{1}{2}p + 1$ is as follows:* $1 + \underbrace{8i - 2}_{1 \leq i \leq \frac{1}{2}p - 1} + (\frac{7}{2}p - 3) + \frac{5}{2}p.$

Proof. The claim is trivially true for levels 0 and 1. For $i \geq 2$, the set of vertices at the i-th level is equal to $S \cup T$, where S and T are as follows:

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Table 4: Set S of vertices at a distance of i from 0 in C_{p+3}, $0 \le i \le \frac{1}{2}p + 1$

S	if	$|S|$
$\{0\}$	$i = 0$	1
$\{i, p+3-i\}$	$1 \le i \le \frac{1}{2}p + 1$ (i even or odd)	2

- $S = \{(x,y) \mid d_{M_p}(x,0) = i$ and $d_{C_{p+3}}(y,0) = j \le i$, where i and j are of the same parity$\}$, and

- $T = \{(x,y) \mid d_{M_p}(x,0) = j < i$ and $d_{C_{p+3}}(y,0) = i$, where i and j are of the same parity$\}$.

Note that $S \cap T = \emptyset$. We distinguish three cases: (i) $2 \le i \le \frac{1}{2}p - 1$, (ii) $i = \frac{1}{2}p$ and (iii) $i = \frac{1}{2}p + 1$.

i. Let $2 \le i \le \frac{1}{2}p - 1$.

   a. Let i be even. The cumulative number of vertices in C_{p+3} at an even distance of $0, 2, 4, \ldots, i$ from 0 is equal to $1 + 2 \times \frac{1}{2}i = i + 1$, so $|S| = 4(i+1)$. Next, the cumulative number of vertices in M_p at an even distance of $0, 2, 4, \ldots, i-2$ from 0 is equal to $1 + 4 \times \frac{1}{2}(i-2) = 2i - 3$, so $|T| = (2i-3) \times 2 = 4i - 6$. Accordingly, $|S| + |T| = 8i - 2$.

   b. Let i be odd. The cumulative number of vertices in C_{p+3} at an odd distance of $1, 3, 5, \ldots, i$ is equal to $2 \times \frac{1}{2}(i+1) = i + 1$, so $|S| = 4(i+1)$. Next, the cumulative number of vertices in M_p at an odd distance of $1, 3, 5, \ldots, i-2$ form 0 is equal to $3 + 4 \times \frac{1}{2}(i-3) = 2i - 3$, so $|T| = (2i-3) \times 2 = 4i - 6$. Accordingly, $|S| + |T| = 8i - 2$.

ii. Let $i = \frac{1}{2}p$ that is even. The number of vertices in M_p at an even distance of $\frac{1}{2}p$ from 0 is equal to 3 (that is the number of vertices at Level 1), while the cumulative number of vertices in C_{p+3} at an even distance of up to $\frac{1}{2}p$ from 0 is equal to $1 + \frac{1}{2}p$, so $|S| = 3(1 + \frac{1}{2}p) = \frac{3}{2}p + 3$. Next, the cumulative number of vertices in M_p at an even distance of up to $\frac{1}{2}p - 2$ from 0 is equal to $1 + 4 \times \frac{1}{2}(\frac{1}{2}p - 2) = p - 3$, so $|T| = 2(p-3)$. Accordingly, $|S| + |T| = \frac{7}{2}p - 3$.

iii. Let $i = \frac{1}{2}p + 1$ that is odd. In this case $|S| = 1 \times (\frac{1}{2}p + 2)$ and $|T| = (p-1) \times 2$, so $|S| + |T| = \frac{5}{2}p$.∎

Theorem 3.2 *The average distance of $M_p \times C_{p+3}$ relative to its order is equal to $\frac{1}{12(p+3)}(p + 2)(4p + 13)$.*

Proof. The average distance (cf. Lemma 3.1) is given by

$$
\begin{aligned}
& \tfrac{1}{p(p+3)}\left(1\times 0 + (\textstyle\sum_{i=1}^{\frac{1}{2}p-1}(8i-2)i) + (\tfrac{7}{2}p-3)\tfrac{1}{2}p + \tfrac{5}{2}p(\tfrac{1}{2}p+1)\right)\\
=\; & \tfrac{1}{p(p+3)}\left(8(\textstyle\sum_{i=1}^{\frac{1}{2}p-1}i^2) - 2(\textstyle\sum_{i=1}^{\frac{1}{2}p-1}i) + 3p^2 + p\right)\\
=\; & \tfrac{1}{12p(p+3)}p(p+2)(4p+13)\\
=\; & \tfrac{1}{12(p+3)}(p+2)(4p+13).\blacksquare
\end{aligned}
$$

Observe that the average-distance$(M_p \times C_{p+3})$ is equal to about two-thirds of diam$(M_p \times C_{p+3}) = \tfrac{1}{2}p + 1$.

4 A routing scheme via shortest paths

The objective of this section is to build a shortest path from the fixed node $(0,0)$ to a node (r,s) in $M_p \times C_{p+3}$, where r and s are not both 0. By Prop. 1.7, the shortest distance depends on the relative parities of r and s.

Algorithm 1 traces a shortest even path and a shortest odd path from 0 to r in M_p. There are two major cases: $0 \le r \le \tfrac{1}{2}p$ and $\tfrac{1}{2}p + 1 \le r \le p - 1$. (If $r = 0$, then a shortest even path consists of node 0 itself, and a shortest odd path consists of the canonical cycle.) For each r, the sum of the length of a shortest 0-to-r even path and the length of a shortest 0-to-r odd path is equal to the odd girth, i.e., $\tfrac{1}{2}p + 1$. See Fig. 3(a) for an illustration.

Algorithm 2 corresponds to C_{p+3}. See Fig. 3(b) for an illustration. In this case, the sum of the length of a shortest even path and the length of a shortest odd path is equal to $p + 3$.

Fact 1 *Let $u_1 - u_2 - \ldots - u_a$ be a 0-to-r path in M_p, and let $v_1 - v_2 - \ldots - v_b$ be a 0-to-s path in C_{p+3}, where a and b are of the same parity, and $u_1 = 0$, $u_a = r$, and $v_1 = 0$, $v_b = s$. If $a = b$, then $(u_1, v_1) - (u_2, v_2) - \ldots - (u_a, v_a)$ is a valid path from $(0,0)$ to (r,s) in $M_p \times C_{p+3}$. If $a < b$, then append the preceding path by $\underbrace{(u_{a-1}, v_{a+1}) - (u_a, v_{a+2}) - \ldots - (u_a, v_b)}$, in which u_{a-1} and u_a alternate as the first co-ordinate. The situation is similar if $a > b$.∎*

Theorem 4.1 *Algorithm 3 correctly traces a shortest $(0,0)$-to-(r,s) path in $M_p \times C_{p+3}$, where $p \equiv 0 \pmod{4}$.*

Proof. By Prop. 1.7.∎

Algorithm 1 A shortest even/odd path in M_p, $p \equiv 0 \pmod 4$

Ensure: $0 \leq r \leq p - 1$
Require: Trace a shortest even path and a shortest odd path from 0 to r
    if $0 \leq r \leq \frac{1}{2}p$ **then**
        if r is even **then**
            a shortest even path is $0 - 1 - 2 - \ldots - r$, and
            a shortest odd path is $0 - (\frac{1}{2}p) - (\frac{1}{2}p - 1) - \ldots - r$;
        else if r is odd **then**
            a shortest even path is $0 - (\frac{1}{2}p) - (\frac{1}{2}p - 1) - \ldots - r$, and
            a shortest odd path is $0 - 1 - 2 - \ldots - r$;
        end if
    end if
    $\triangleright$
    if $\frac{1}{2}p + 1 \leq r \leq p - 1$ **then**
        if r is even **then**
            a shortest even path is $0 - (p - 1) - (p - 2) - \ldots - r$, and
            a shortest odd path is $0 - (\frac{1}{2}p) - (\frac{1}{2}p + 1) - \ldots - r$;
        else if r is odd **then**
            a shortest even path is $0 - (\frac{1}{2}p) - (\frac{1}{2}p + 1) - \ldots - r$, and
            a shortest odd path is $0 - (p - 1) - (p - 2) - \ldots - r$;
        end if
    end if
---

5   Jump sequence associated with $M_p \times C_{p+3}$

This section determines the jump sequence associated with the circulants.

Theorem 5.1 *[20] If $gcd(r, t) = 1$, then $C_r \times C_t$ admits a Hamiltonian cycle.*

Proof. The following sequence of vertices in $C_r \times C_t$ corresponds to a Hamiltonian cycle: $x_0, \ldots, x_{rt-1}$, where $x_k = (k \bmod r, k \bmod t)$. Intuitively, the cycle is obtainable as follows. Start at $(0, 0)$, and at each step, increment the first co-ordinate modulo r and simultaneously increment the second co-ordinate modulo t. Fig. 5 illustrates the construction in respect of $C_8 \times C_5$.∎

Note: There exists a result more general than Theorem 5.1, viz., $C_r \times C_t$ is Hamiltonian decomposable iff r and t are not both even [16].

Theorem 5.2 *If $p \equiv 4 \pmod{12}$, then $M_p \times C_{p+3} \cong \mathcal{C}_{p(p+3)}(1, s, 2s + 1)$, where $s = \frac{1}{6}(p + 2)(p + 3) - 1$.*

Algorithm 2 A shortest even/odd path in C_{p+3}, $p \equiv 0 \pmod 4$

Ensure: $0 \le s \le p + 2$

Require: Trace a shortest even path and a shortest odd path from 0 to s

   if s is even **then**

      a shortest even path is $0 - 1 - 2 - \ldots - s$, and

      a shortest odd path is $0 - (p + 2) - (p + 1) - \ldots - s$;

   else if s is odd **then**

      a shortest even path is $0 - (p + 2) - (p + 1) - \ldots - s$, and

      a shortest odd path is $0 - 1 - 2 - \ldots - s$;

   end if

Algorithm 3 A shortest $(0,0)$-to-(r, s) path in $M_p \times C_{p+3}$, $p \equiv 0 \pmod 4$

Ensure: $0 \le r \le p - 1$ and $0 \le s \le p + 2$.

Require: r and s are not both 0.

1: invoke Algorithm 1 to obtain a shortest 0-to-r even path, say, Π_0 and a shortest 0-to-r odd path, say, Π_1, both in M_p;

2: invoke Algorithm 2 to obtain a shortest 0-to-s even path, say, Δ_0 and a shortest 0-to-s odd path, say, Δ_1, both in C_{p+3};

3: employ Fact 1 to build a $(0,0)$-to-(r, s) path, say, Π in $M_p \times C_{p+3}$ using Π_0 and Δ_0;

4: employ Fact 1 to build a $(0,0)$-to-(r, s) path, say, Δ in $M_p \times C_{p+3}$ using Π_1 and Δ_1;

5: **return** shorter of the two paths Π and Δ;

Proof. First recall that ± 1 and $\pm\frac{1}{2}p$ are the jumps associated with the circulant M_p, and ± 1 are the jumps associated with the circulant C_{p+3}. Next, let $x_0, \cdots, x_{p(p+3)-1}$ be the sequence of vertices corresponding to a Hamiltonian cycle in $M_p \times C_{p+3}$ that is based on the proof of Theorem 5.1. It suffices to show that each of $\{x_k, x_{k+s}\}$ and $\{x_k, x_{k+2s+1}\}$ is in $E(M_p \times C_{p+3})$, $0 \le k \le p(p+3) - 1$, where $k + s$ and $k + 2s + 1$ are each taken modulo $p(p + 3)$.

Let $x_k = (i, j)$, so $x_{k+s} = (i + s, j + s)$ and $x_{k+2s+1} = (i + 2s + 1, j + 2s + 1)$, where the arithmetic is modulo p in the first co-ordinate, and modulo $p + 3$ in the second. The following statements ensure the membership of each of $\{i, i + s\}$ and $\{i, i + 2s + 1\}$ in $E(M_p)$. (Verification is left to the reader.)

- $s = \frac{1}{6}(p + 2)(p + 3) - 1 \equiv \frac{1}{2}p \pmod p$, and

- $2s + 1 = \frac{1}{3}(p + 2)(p + 3) - 1 \equiv 1 \pmod p$.

Next, the statements below show that $\{j, j + s\}$, $\{j, j + 2s + 1\} \in E(C_{p+3})$:

- $s = \frac{1}{6}(p + 2)(p + 3) - 1 \equiv -1 \pmod{p + 3}$, and

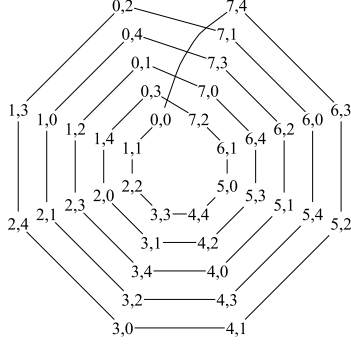- $2s + 1 = \frac{1}{3}(p + 2)(p + 3) - 1 \equiv -1 \pmod{p + 3}$.

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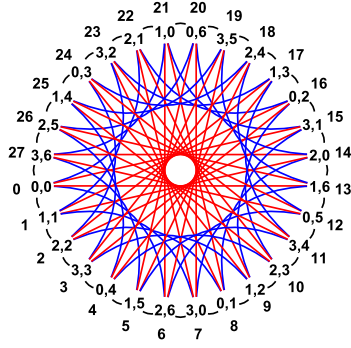Figure 5: A Hamiltonian cycle in $C_8 \times C_5$

Figure 6: $M_4 \times C_7 \cong \mathcal{C}_{28}(1, 6, 13)$

It follows that each of $\{x_k, x_{k+s}\}$ and $\{x_k, x_{k+2s+1}\}$ is in $E(M_p \times C_{p+3})$. Fig. 6 illustrates the proof in respect of $M_4 \times C_7 \cong \mathcal{C}_{28}(1, 6, 13)$.∎

Theorem 5.3 *If* $p \equiv 8 \pmod{12}$, *then* $M_p \times C_{p+3} \cong \mathcal{C}_{p(p+3)}(1, s, 2s - 1)$, *where* $s = \frac{1}{6}(p - 2)(p + 3) + 1$.

Proof. The argument is similar to that in the proof of Theorem 5.2. The essential facts are as follows:

- $s = \frac{1}{6}(p - 2)(p + 3) + 1 \equiv \frac{1}{2}p \pmod{p}$, and
- $2s - 1 = \frac{1}{3}(p - 2)(p + 3) + 1 \equiv -1 \pmod{p}$;

and

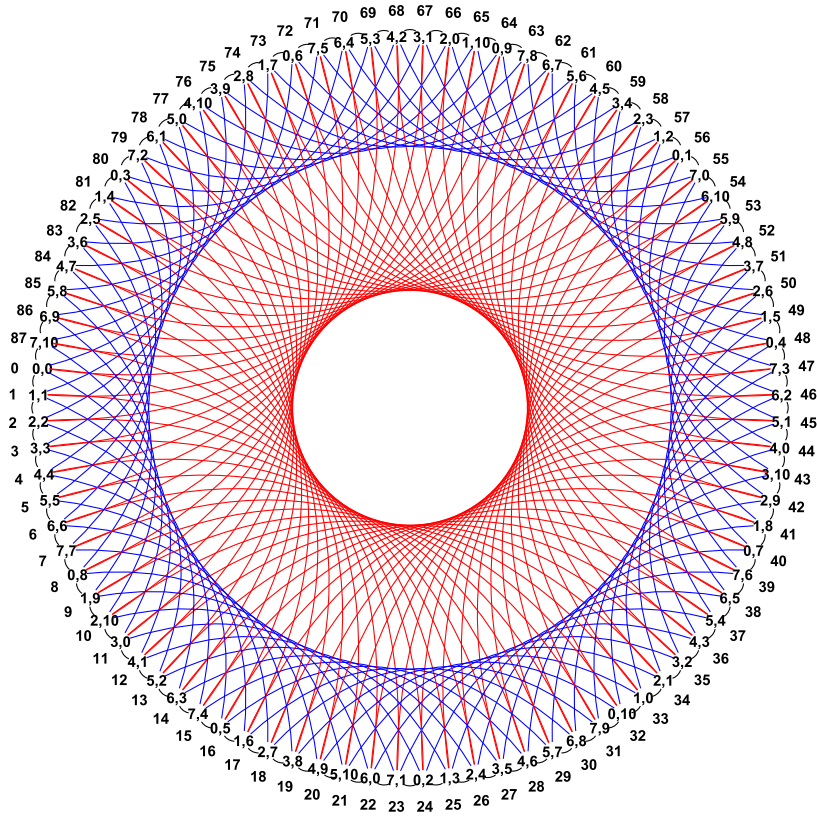- $s = \frac{1}{6}(p - 2)(p + 3) + 1 \equiv 1 \pmod{p + 3}$, and

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Figure 7: $M_8 \times C_{11} \cong \mathcal{C}_{88}(1, 12, 23)$

- $2s - 1 = \frac{1}{3}(p-2)(p+3) + 1 \equiv 1 \pmod{p+3}$.

Fig. 7 illustrates the proof in respect of $M_8 \times C_{11} \cong \mathcal{C}_{88}(1, 12, 23)$. ∎

Remark: By Lemma 2.1, $C_{\frac{1}{2}p+1} \times C_{p+3}$ is an isometric subgraph of $M_p \times C_{p+3}$. Interestingly enough, $C_{\frac{1}{2}p+1} \times C_{p+3}$ itself is a circulant. More precisely, $C_{\frac{1}{2}p+1} \times C_{p+3} \cong \mathcal{C}_{(\frac{1}{2}p+1)(p+3)}(1, 2p+5)$ [21]. In that light, the six-regular circulant $M_p \times C_{p+3}$ may be viewed as an extension of the four-regular circulant $C_{\frac{1}{2}p+1} \times C_{p+3}$. The odd girth, diameter and chromatic number of $M_p \times C_{p+3}$ are equal to the respective parameters of $C_{\frac{1}{2}p+1} \times C_{p+3}$.

The following result adds a distinctive property to the graphs.

Theorem 5.4 *The circulants appearing in Theorems 5.2 and 5.3 are each Hamiltonian decomposable, and their vertex connectivity is equal to six.*

Proof. Dean [12] proved that a six-regular circulant on n vertices admits a Hamiltonian decomposition if one of its step sizes corresponds to an element of order n. It is clear that each circulant in Theorem 5.2 as well as 5.3 includes the step size of one that is of order n.

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That the vertex connectivity is equal to six follows from a theorem by Boesch and Felzer [4] relating to a six-regular circulant, one of whose step sizes is equal to one.∎

6 Embedding of $M_p \times C_{p+3}$ on a twisted torus

This section presents an algorithm to embed $M_p \times C_{p+3}$ on a torus with a half twist. (See Algorithm 4.) This kind of placement has certain advantages. As noted by Sequin [28], for instance, embedding on a (twisted) torus means homogeneous multiprocessor configuration without boundaries. In the present embedding, every processor has six nearest neighbors, and connections between all pairs of processors are approximately of an equal length. In a related development, Cámara *et al.* [8] presented twisted torus topologies derived from the Cartesian product [16] of up to three cycles.

Lemma 6.1 *Algorithm 4 correctly builds the embedding of $M_p \times C_{p+3}$.*

Proof. First consider Step 1, and note that $\phi(i_1, j_1) \neq \phi(i_2, j_2)$ if $j_1 \neq j_2$, i.e., nodes not in the same row are distinct. Next, p and $\frac{1}{2}p - 1$ being coprime, $(\frac{1}{2}p - 1)i_1 \not\equiv (\frac{1}{2}p - 1)i_2$ (mod p) if $i_1 \neq i_2$, i.e., nodes in the j-th row are pairwise distinct if j is even. Similarly, nodes in the j-th row are distinct if j is odd. It follows that the vertices on the array at Step 1 are well-defined, mutually distinct and exhaustive with respect to $V(M_p \times C_{p+3})$.

For validity of the three kinds of edges at Step 2, it suffices to show that the first coordinate of $\phi(i, j)$ is adjacent to the first coordinate of each of $\phi(i, j + 1)$, $\phi(i + 1, j + 1)$ and $\phi(i - 1, j + 1)$ in M_p. To that end, we prove that the absolute difference between the respective coordinates taken modulo p is equal to 1, $p - 1$ or $\frac{1}{2}p$. The statements below establish the claim for even j. (The case for odd j is similar.) The correctness itself is based on the following fact: If x, y and n are integers, where n is positive and $0 < |x - y| < n$, then $|(x \bmod n) - (y \bmod n)| \bmod n = |x - y|$ or $n - |x - y|$.

a. $\left| ((\frac{1}{2}p - 1)i \bmod p) - (((\frac{1}{2}p - 1)i + \frac{1}{2}p) \bmod p) \right| \bmod p = \frac{1}{2}p$.

b. $\left| ((\frac{1}{2}p - 1)i \bmod p) - (((\frac{1}{2}p - 1)(i + 1) + \frac{1}{2}p) \bmod p) \right| \bmod p = 1$ or $p - 1$.

c. $\left| ((\frac{1}{2}p - 1)i \bmod p) - ((\frac{1}{2}p - 1)(i - 1) + \frac{1}{2}p \bmod p) \right| = 1$ or $p - 1$.

Check to see that the "latitudinal wrap-around" edges introduced at Step 5, and the "longitudinal wrap-around" edges introduced at Step 7 are well-defined. Further, the sets of edges at Steps 3, 5 and 7 are mutually disjoint. Accordingly, the cumulative number of edges is equal to $(3p^2 + 4p - 4) + (2p + 4) + 3p = 3p(p + 3)$ that is equal to $|E(M_p \times C_{p+3})|$.∎

Notice that the torus in the present case involves a half twist, i.e., a twist by 180°. This may be seen at Step 7 of Algorithm 4. The graph itself is such that the node $(i, 0)$ is not adjacent to $(i, p + 2)$, rather $(i, 0)$ is adjacent to $(i + \frac{1}{2}p, p + 2)$. In the process, a half

Algorithm 4 Embedding of $M_p \times C_{p+3}$, $p \equiv 0 \pmod 4$

1: Lay out the vertices of $M_p \times C_{p+3}$ on a $(p+3) \times p$ rectangular array in which the (i, j)-th position is occupied by the vertex $\phi(i, j)$, where

$$\phi(i, j) = \begin{cases} \left((\frac{1}{2}p - 1)i \bmod p, \ j \right) & \text{if } j \text{ is even} \\ \left((\frac{1}{2}p - 1)i + \frac{1}{2}p \bmod p, \ j \right) & \text{if } j \text{ is odd} \end{cases}$$

$0 \le i \le p - 1$, $0 \le j \le p + 2$.

2: ▷ The vertices at the four "corners" are: $\phi(0, 0) = (0, 0)$, $\phi(p - 1, 0) = (\frac{1}{2}p + 1, 0)$, $\phi(0, p + 2) = (0, p + 2)$ and $\phi(p - 1, p + 2) = (\frac{1}{2}p + 1, p + 2)$.

3: Introduce the following:

- "Vertical" edges: $\{\phi(i, j), \phi(i, j + 1)\}$, $0 \le i \le p - 1$, $0 \le j \le p + 1$

- "Diagonal" edges: $\{\phi(i, j), \phi(i + 1, j + 1)\}$, $0 \le i \le p - 2$, $0 \le j \le p + 1$

- "Reverse diagonal" edges: $\{\phi(i, j), \phi(i - 1, j + 1)\}$, $0 \le i \le p - 2$, $1 \le j \le p + 2$.

4: ▷ The number of edges introduced at Step 3 is $3p^2 + 4p - 4$. Fig. 8 illustrates the construction so far in respect of $M_8 \times C_{11}$, where the three types of edges appear in blue, red and green, respectively.

5: Introduce the following "latitudinal wrap-around" edges:

- $(0, 0) - (1, 1) - (0, 2) - (1, 3) - \cdots - (1, p + 1) - (0, p + 2)$, and

- $(\frac{1}{2}p + 1, 0) - (\frac{1}{2}p, 1) - (\frac{1}{2}p + 1, 2) - (\frac{1}{2}p, 3) - \cdots - (\frac{1}{2}p, p + 1) - (\frac{1}{2}p + 1, p + 2)$.

6: ▷ The number of edges introduced at Step 5 is $2p + 4$. The resulting "cylinder" appears in Fig. 9 in respect of $M_8 \times C_{11}$, where the "latitudinal wrap-around" edges appear in "dotted pink." For the sake of clarity, the "hidden" vertices have not been shown. (Ignore the "thin dotted lines.")

7: Introduce the "longitudinal wrap-around" edges $\{i, 0), (i - 1, p + 2)\}$, $\{i, 0), (i + 1, p + 2)\}$ and $\{i, 0), (i + \frac{1}{2}p, p + 2)\}$ for $i = 0, \ldots, p - 1$, where $i - 1$, $i + 1$ and $i + \frac{1}{2}p$ are each modulo p.

8: ▷ The edges introduced at Step 7 total $3p$. Fig. 10 illustrates them in respect of $M_8 \times C_{11}$. (Ignore the "thin dotted lines.").

9: ▷ The final form of $M_p \times C_{p+3}$ appears in Fig. 11.

Figure 8: Construction at the end of Step 3 in respect of $M_8 \times C_{11}$

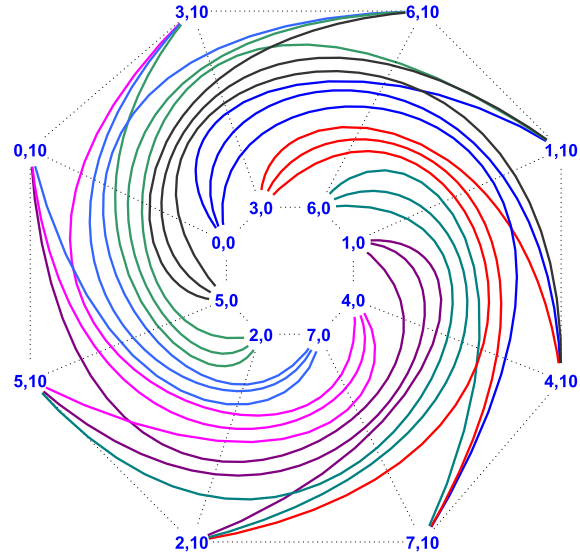Figure 9: Construction at the end of Step 5 in respect of $M_8 \times C_{11}$

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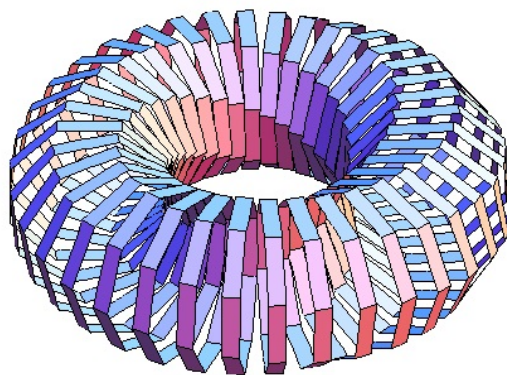Figure 10: The "longitudinal" wrap-around edges in $M_8 \times C_{11}$

Figure 11: Final form of $M_p \times C_{p+3}$

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twist becomes necessary. Note also that there are certain edge crossings in the embedding. Indeed, an edge is crossed by at most one other edge. This kind of configuration has been called an immersion in the literature [23].

7   Concluding remarks

This paper introduces a family of three-colorable, six-regular circulants representable as the Kronecker product of a Möbius ladder and an odd cycle. The order of each graph is $4d^2 - 2d - 2$, where d is its diameter, and $d \equiv 3, 5 \pmod 6$, cf. Theorem 2.5(2). Interestingly enough, the circulants inherit the twist existing in the Möbius ladder in that they admit an embedding on a torus with a half twist.

A family of six-regular circulants, called triple-loop networks, has been studied at great lengths in the literature [35, 9, 30, 31]. Its order is $3d^2 + 3d + 1$, where d is its diameter. See Table 1 in Sec. 1 for a comparison.

Efficient gossiping, efficient broadcasting and superior routing are some of the problems associated with the circulants in this paper that merit attention.

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