Hamiltonian Decomposition of the Rectangular Twisted Torus

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Abstract—We show that the $2a \times a$ rectangular twisted torus introduced by Cámara et al. [5] is edge decomposable into two Hamiltonian cycles. In the process, the $2a \times a \times a$ prismatic twisted torus is edge decomposable into three Hamiltonian cycles, and the $2a \times a \times a$ prismatic doubly twisted torus admits two edge-disjoint Hamiltonian cycles.

Index Terms—Graphs and networks, rectangular twisted torus, Hamiltonian decomposition, multiprocessor interconnection, fault tolerance.

1 Introduction

Cámara et al. [5] introduced the $2a \times a$ *rectangular twisted torus* (RTT) as an alternative to the $2a \times a$ *torus*, the latter being representable as the *Cartesian product* of the *cycles* C_{2a} and C_a, where $a \geq 3$. Further, they extended the concept to the three dimensions and presented the $2a \times a \times a$ *prismatic twisted torus* (PTT) and the $2a \times a \times a$ *prismatic doubly twisted torus* (PDTT). The motivation comes from a potential improvement to the existing topologies built around a 2D/3D torus currently in use by HP, Cray, IBM, etc. [5].

We prove that the $2a \times a$ RTT has the welcome property of being *Hamiltonian decomposable*. The result leads to an analogous decomposition of the PTT and a set of two edge-disjoint *Hamiltonian cycles* in the PDTT.

1.1 Preliminaries

When we speak of a graph, we mean a finite, simple, undirected, and connected graph. Let P_m denote a *path* on m vertices, and let C_n denote a *cycle* on n vertices, where $m \geq 1$ and $n \geq 3$, and where adjacencies exist in the natural way. The Cartesian product $G \square H$ of graphs $G = (V, E)$ and $H = (W, F)$ is defined as follows: $V(G \square H) = V \times W$ and $E(G \square H) = \{\{(a, x), (b, y)\}: \{a, b\} \in E \text{ and } x = y, \text{ or } \{x, y\} \in F \text{ and } a = b\}$. For missing definitions and details, see Hammack et al. [8].

A simple spanning cycle, if one exists, in a graph is called a Hamiltonian cycle. Determining the existence of such a cycle is known to be NP-hard.

A graph is said to be Hamiltonian decomposable if it is regular of degree, say, Δ and if its edge set admits a partition into $\Delta/2$ Hamiltonian cycles when Δ is even, or into $(\Delta - 1)/2$ Hamiltonian cycles and a perfect matching when Δ is odd. Not every regular graph has this property.

For $a \geq 3$, the $2a \times a$ RTT is a four-regular graph on vertices (i, j): $0 \leq i \leq 2a - 1$ and $0 \leq j \leq a - 1$, where

- $(i, 0)$ is adjacent to each of $(i + 1, 0)$, $(i - 1, 0)$, $(i, 1)$, and $(i + a, a - 1)$.
- For $1 \leq j \leq a - 2$, (i, j) is adjacent to each of $(i + 1, j)$, $(i - 1, j)$, $(i, j + 1)$, and $(i, j - 1)$, and
- $(i, a - 1)$ is adjacent to each of $(i + 1, a - 1)$, $(i - 1, a - 1)$, $(i, a - 2)$, and $(i + a, 0)$.

The arithmetic is modulo $2a$ in the first coordinate and modulo a in the second. Fig. 1 depicts the 8×4 RTT.

The $2a \times a \times a$ PTT is given by the Cartesian product of the $2a \times a$ RTT and the cycle C_a. For the $2a \times a \times a$ PDTT, let G be the Cartesian product of the $2a \times a$ RTT and the path P_a, and introduce the matching $\{\{(x, y, 0), (x + a, y, a - 1)\} : 0 \leq x \leq 2a - 1, \ 0 \leq y \leq a - 1\}$ that is of size $2a^2$ to G. What results is the $2a \times a \times a$ PDTT.

Each of the $2a \times a \times a$ PTT and $2a \times a \times a$ PDTT may be viewed as consisting of $2a$ isomorphic layers along x-axis (resp. a isomorphic layers along y-axis, or a isomorphic layers along z-axis). Figs. 2 and 3 depict them in respect of the $8 \times 4 \times 4$ PTT and $8 \times 4 \times 4$ PDTT, respectively. See Cámara et al. [5] for three-dimensional views.

1.2 Hamiltonian Decomposable Graphs

Hamiltonian decomposable graphs are amenable to numerous applications in engineering and computer science. Accordingly, they have a rich literature. Some of their major applications are as follows:

1. A Hamiltonian decomposable graph has a high degree of connectivity, so it can tolerate a large number of node failures and/or edge failures; in particular, a simple route may exist between two nodes in the healthy portion of the graph [6], [9].
2. There is a substantial improvement in the efficiency of the data transmission, since a message may be broken down into smaller pieces and sent along the edge-disjoint cycles. Accordingly, problems involving sorting, many-to-many broadcasting, and many-to-many scattering may be nearly optimally solved using parallel algorithms on a graph that is Hamiltonian decomposable [6], [9].

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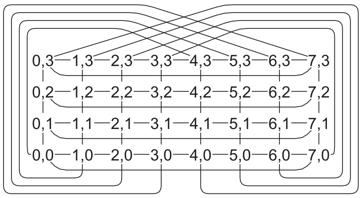

Fig. 1. The 8×4 rectangular twisted torus.

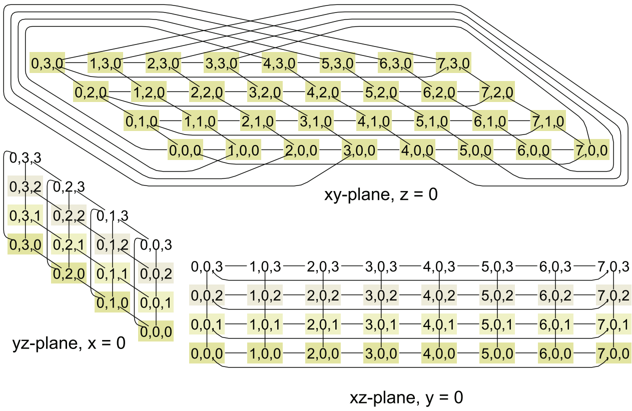

Fig. 2. Layers of the $8 \times 4 \times 4$ PTT in the three planes.

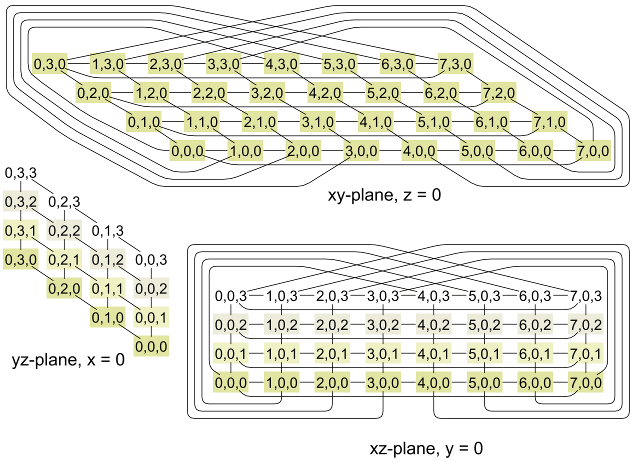

Fig. 3. Layers of the $8 \times 4 \times 4$ PDTT in the three planes.

3. Routing balanced communication demands is very efficient in an all-optical network that is Hamiltonian decomposable [13].

It is no coincidence, therefore, that most networks currently in use, e.g., hypercubes, tori, butterflies, etc., are Hamiltonian decomposable [7], [9].

1.3 Related Work

Bosák [3] contains Hamiltonian decomposition of the torus (p. 128) and other graphs like hypercubes, complete graphs, and products of cycles. Our proof in respect of the RTT is similar to his proof in respect of the torus.

In a quest for better topologies, a twist was introduced in one of the two dimensions of a 2D torus in ILLIAC IV [4] in early 1970's, leading to a better performance. See Martin [11], Sequin [12] and Beivide et al. [2] for subsequent studies. A recent application of a twist in a torus exists in HP GS1280 computer [6]. See Yang et al. [15] for similar

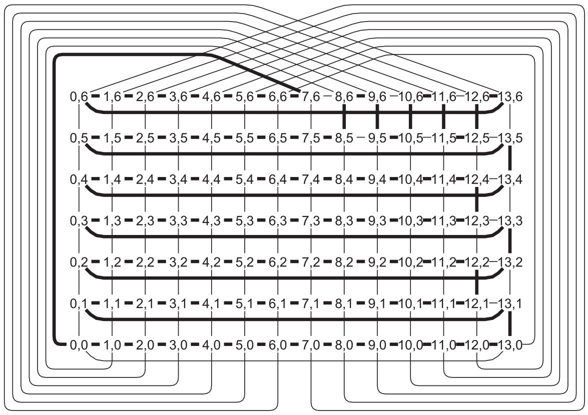

Fig. 4. Hamiltonian decomposition of the 14×7 rectangular twisted torus.

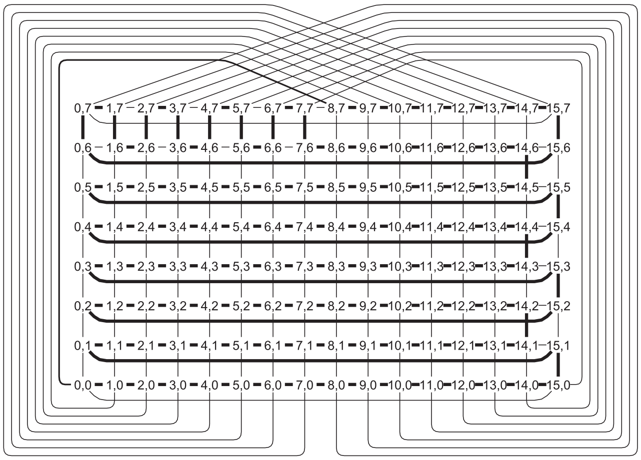

Fig. 5. Hamiltonian decomposition of the 16×8 rectangular twisted torus.

topologies. No previous work is known about these graphs admitting Hamiltonian decomposition.

In a related study, Yang et al. [14] built locally twisted cubes as a variant of the hypercube structure. Among various advantages, the resulting network admits two edge-disjoint Hamiltonian cycles [10]. It is not clear at this point whether or not the graph is Hamiltonian decomposable.

2 Result

The following is our central result:

Theorem 1. *For $a \geq 3$, the $2a \times a$ RTT admits an edge decomposition into two Hamiltonian cycles.*

Proof. For $a = 3, 4, 5$, and 6, Hamiltonian decompositions of the $2a \times a$ RTT appear in the *supplementary file*, which can be found on the Computer Society Digital Library at http://doi.ieeecomputersociety.org/10.1109/TPDS.2011.288. In what follows, let a be greater than or equal to 7.

We distinguish two cases: a odd and a even. See Figs. 4 and 5, which depict the Hamiltonian decompositions of the 14×7 RTT and the 16×8 RTT, respectively. The two provide a method for constructing a Hamiltonian decomposition in the general case for odd a and even a, respectively. Meanwhile Table 1 presents the sequence of vertices corresponding to the first Hamiltonian cycle (with thick edges in Fig. 4) in the $2a \times a$ RTT for a odd,

TABLE 1
Sequence (i_k, j_k): $0 \le k \le 2a^2-1$ for the First Hamiltonian Cycle in the $2a \times a$ RTT, a Odd and $a \ge 7$

for $0 \le k \le 2a-1$: $i_k = k$;	for $0 \le k \le 6a-1$: $j_k = k/(2a)$;
$i_{2a} = 2a-1$;	for $6a \le k \le 2a^2-4a+3$: $j_k = j_{k-4a}+2$;
for $2a+1 \le k \le 4a-1$: $i_k = k-(2a+1)$;	for $2a^2-4a+2 \le k \le 2a^2-3a+2$: $j_k = j_{k-1}$;
$i_{4a} = 2a-2$;	for $k = 2a^2-3a+3$: $j_k = j_{k-1}+1$;
for $4a+1 \le k \le 6a-2$: $i_k = i_{k-1}-1$;	for $k = 2a^2-3a+4$: $j_k = j_{k-1}$;
$i_{6a-1} = 2a-1$; $i_{6a} = 2a-1$;	for $k = 2a^2-3a+5$: $j_k = j_{k-1}-1$;
for $6a+1 \le k \le 2a^2-4a+3$: $i_k = i_{k-4a}$;	for $2a^2-3a+6 \le k \le 2a^2-(a+2)$: $j_k = j_{k-4}$;
for $2a^2-4a+2 \le k \le 2a^2-3a+2$: $i_k = i_{k-1}+1$;	for $k = 2a^2-(a+1)$: $j_k = j_{k-1}$;
for $k = 2a^2-3a+3$: $i_k = i_{k-1}$;	for $2a^2-a \le k \le 2a^2-1$: $j_k = j_{k-1}$;
for $k = 2a^2-3a+4$: $i_k = i_{k-1}+1$;	
for $k = 2a^2-3a+5$: $i_k = i_{k-1}$;	
for $2a^2-3a+6 \le k \le 2a^2-(a+2)$: $i_k = i_{k-4}+2$;	
for $k = 2a^2-(a+1)$, $i_k = 0$;	
for $2a^2-a \le k \le 2a^2-1$: $i_k = i_{k-1}+1$;	

TABLE 2
Sequence (i_k, j_k): $0 \le k \le 2a^2-1$ for the First Hamiltonian Cycle in the $2a \times a$ RTT, a Even and $a \ge 8$

for $0 \le k \le 2a-1$: $i_k = k$;	for $0 \le k \le 6a-1$: $j_k = k/(2a)$;
$i_{2a} = 2a-1$;	for $6a \le k \le 2a^2-(3a+1)$: $j_k = j_{k-4a}+2$;
for $2a+1 \le k \le 4a-1$: $i_k = k-(2a+1)$;	for $k = 2a^2-3a$: $j_k = j_{k-1}+1$;
$i_{4a} = 2a-2$;	for $k = 2a^2-3a+1$: $j_k = j_{k-1}$;
for $4a+1 \le k \le 6a-2$: $i_k = i_{k-1}-1$;	for $k = 2a^2-3a+2$: $j_k = j_{k-1}-1$;
$i_{6a-1} = 2a-1$; $i_{6a} = 2a-1$;	for $2a^2-3a+3 \le k \le 2a^2-(a+2)$: $j_k = j_{k-4}$;
for $6a+1 \le k \le 2a^2-(3a+1)$: $i_k = i_{k-4a}$;	for $k = 2a^2-(a+1)$: $j_k = j_{k-1}$;
for $k = 2a^2-3a$: $i_k = a-1$;	for $k = 2a^2-a$: $j_k = j_{k-1}+1$;
for $k = 2a^2-3a+1$: $i_k = a-2$;	for $2a^2-(a-1) \le k \le 2a^2-1$: $j_k = j_{k-1}$;
for $k = 2a^2-3a+2$: $i_k = a-2$;	
for $2a^2-3a+3 \le k \le 2a^2-(a+2)$: $i_k = i_{k-4}-2$;	
for $k = 2a^2-(a+1)$: $i_k = 2a-1$;	
for $k = 2a^2-a$: $i_k = 2a-1$;	
for $2a^2-(a-1) \le k \le 2a^2-1$: $i_k = i_{k-1}-1$;	

while Table 2 does that for a even. The corresponding sequences for the second Hamiltonian cycles in the two cases available in the online supplemental material, which also includes computer programs in C++ that generate the sequences for a given $a \ge 7$. $\square$

Corollary 2. *For $a \ge 3$, the $2a \times a \times a$ PTT admits an edge decomposition into three Hamiltonian cycles.*

Proof. It is known that if G is a graph that is edge decomposable into two Hamiltonian cycles, then $G \,\square\, C_n$ is edge decomposable into three Hamiltonian cycles [1]. The result is immediate. $\square$

Corollary 3. *For $a \ge 3$, the $2a \times a \times a$ PDTT contains two edge-disjoint Hamiltonian cycles.*

Proof. Let G denote the $\square$-product of the $2a \times a$ RTT and the path P_a, and note that G consists of a copies of the $2a \times a$ RTT *stacked* together and connected by the *vertical* edges of the form $\{(x, y, k), (x, y, k+1)\}$, that run in the z-direction, $0 \le k \le a-2$. By Theorem 1, each copy consists of two edge-disjoint spanning cycles, say, of color 1 and color 2, respectively. Now carefully splice the cycles of each color separately. To that end, discard a few edges of these cycles that are appropriately positioned and utilize as many *vertical* edges to arrive at two edge-disjoint Hamiltonian cycles in G. The construction is well defined since there is an abundance of vertical edges for that purpose. Further, the claim holds as G itself is a spanning subgraph of the $2a \times a \times a$ PDTT. $\square$

Fig. 6 illustrates the proof of Corollary 3 in respect of the $8 \times 4 \times 4$ PDTT. (The arrowheads on the vertical edges are meant for a quick understanding of the construction.)

In a quest for a Hamiltonian decomposition of the $2a \times a \times a$ PDTT, we stripped it of the two Hamiltonian cycles obtainable from the proof of Corollary 3 to see if the remaining edges constitute another Hamiltonian cycle, but the attack was not successful. Whether or not this graph admits an edge decomposition into (three) Hamiltonian cycles seems to be a very interesting problem.

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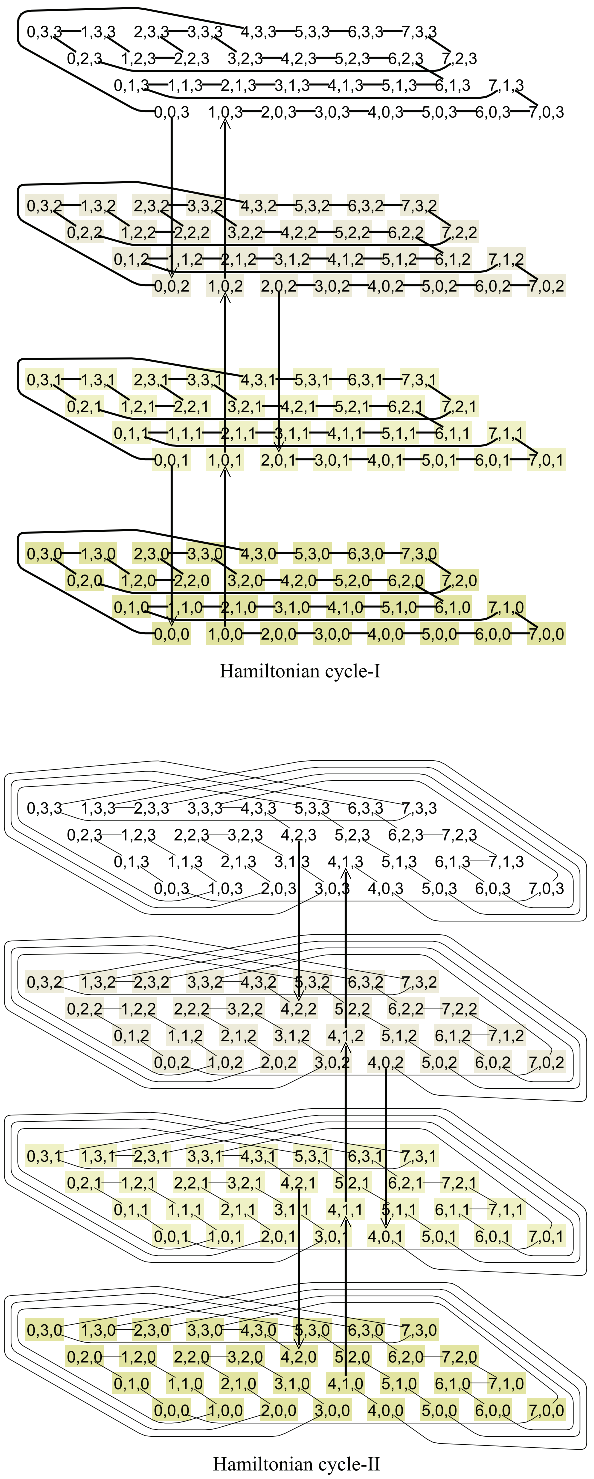

Fig. 6. Two edge-disjoint Hamiltonian cycles in the $8 \times 4 \times 4$ PDTT.

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