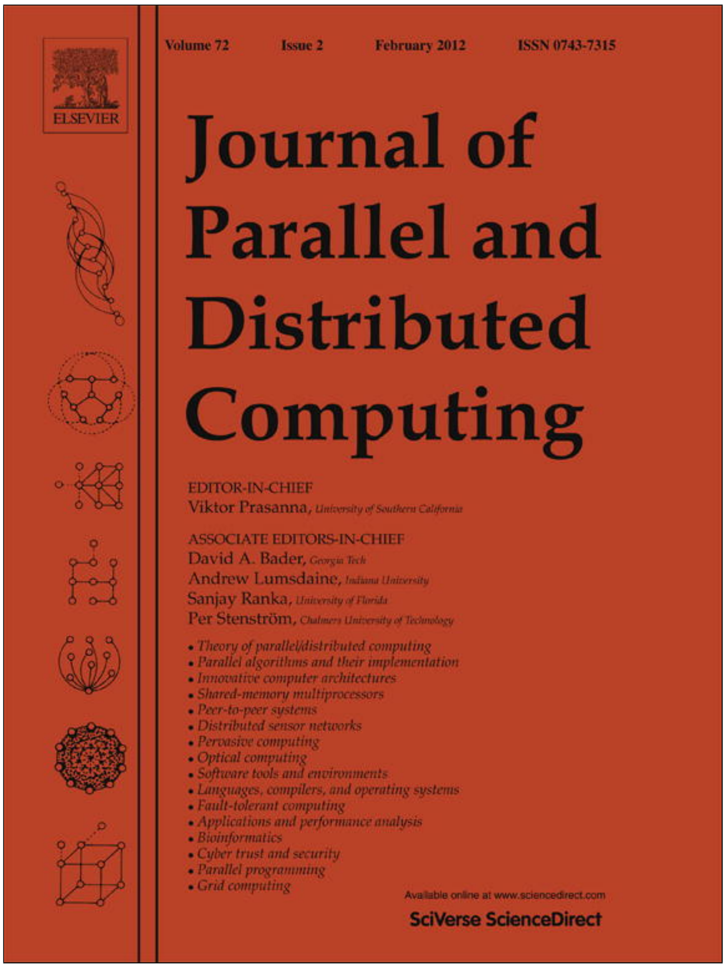
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Orthogonal drawings and crossing numbers of the Kronecker product of two cycles

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ABSTRACT

An orthogonal drawing of a graph is an embedding of the graph in the plane such that each edge is representable as a chain of alternately horizontal and vertical line segments. This style of drawing finds applications in areas such as optoelectronic systems, information visualization and VLSI circuits. We present orthogonal drawings of the Kronecker product of two cycles around vertex partitions of the graph into grids. In the process, we derive upper bounds on the crossing number of the graph. The resulting upper bounds are within a constant multiple of the lower bounds. Unlike the Cartesian product that is amenable to an inductive treatment, the Kronecker product entails a case-to-case analysis since the results depend heavily on the parameters corresponding to the lengths of the two cycles.

1. Introduction and preliminaries

An *orthogonal drawing* of a graph consists of an embedding of the graph in the plane such that each edge is representable as a sequence of alternately horizontal and vertical line segments. This style of drawing finds applications in areas such as optoelectronic systems, information visualization and VLSI circuits [27,35]. The drawing itself is restricted to graphs of maximum vertex degree four.

The *Kronecker product* $C_m \times C_n$ of two *cycles*, which we formally define below, is a four-regular graph with a number of applications in engineering, computer science and related disciplines. For example, if m and n are both odd, then $C_m \times C_n$, which is known as a diagonal mesh [34,17,33], has a lower diameter, higher independence number and higher odd girth relative to its closest rival $C_m \Box C_n$, which is known as a toroidal mesh [16]. Pearlmutter [28] showed that a diagonal mesh is isomorphic to a twisted toroidal mesh and that a twisted toroidal topology was earlier used as the routing network of the FAIM-1 parallel computer [5].

We construct orthogonal drawings of $C_m \times C_n$ with the principal objective of minimizing the number of edge crossings in the embedding. Our method of attack consists of partitioning $C_m \times C_n$ into vertex-disjoint grids that may be viewed as clusters. The edges of the graph not in the grids appear as disjoint matchings, which we carefully introduce within and around the grids. Meanwhile partitioning of $C_m \times C_n$ into grids is a result that is of independent interest by itself.

Whereas an exact value of the *crossing number* of $C_m \times C_n$ is elusive, our drawings lead to upper bounds that are within a constant multiple of the lower bounds. Meanwhile the present paper is the first systematic study in the area of graph drawings and crossing numbers of the Kronecker product. Unlike $C_m \Box C_n$ that is amenable to an inductive treatment, $C_m \times C_n$ entails an analysis on a case-to-case basis, since the results depend heavily on the types of the parameters m and n.

Orthogonal drawing applied to computer systems

An orthogonal drawing promotes optical distinctiveness of the edges incident on a vertex, since the minimum angle between adjacent edges is $\pi/2$. Accordingly, it is the most appealing of all drawing styles. The following are some of the applications of this model to computer science and engineering:

1. Computer hardware and microchips are designed using CAD tools, which must create a layout of the logic gates and their interconnections on circuit boards. The layouts themselves

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correspond to a grid drawing in which all vertices and bends of the edges have integer co-ordinates [7]. An orthogonal drawing comes closest to a grid drawing.
2. The basic abstraction underlying a software system running on a set of distinct machines consists of a set of finite transition systems. An orthogonal drawing gives an intuitive representation of such systems [9].
3. Entity-relationship diagrams are one of the common methods to structure large volumes of data by defining attributes on and relations between the data [9]. Both entities and their attributes are modeled as nodes of a graph. Further, edges exist between entities and their respective attributes, and annotated edges exist among entities depicting certain constraints. The resulting graph is usually presented by means of an orthogonal drawing.

Crossing number

The crossing number of a graph G is denoted by $\mathrm{cr}(G)$ and is defined to be the least number of edge crossings in any drawing of G in the plane. It is an important topological invariant. Here are certain applications of this parameter:

1. It is closely related to a lower bound on the chip area requirements (within technological constraints) for the VLSI circuit layout of the graph [4,24].
2. It is the most important parameter measuring the deviation of the graph from being planar.
3. It plays an important role in various fields of discrete/computational geometry [32].
4. It influences the aesthetics and readability of a graph in automated graph drawing.

The general problem of determining this invariant is NP-hard even for cubic graphs [14]. Even et al. [8] earlier presented an approximation algorithm for $\mathrm{cr}(G) + |V(G)|$ of a bounded-degree graph G. There is an impressive online bibliography of the literature on crossing numbers maintained by Vrt'o [37].

Basic definitions

When we speak of a graph, we mean a finite, simple and undirected graph having at least two vertices. The graphs are also connected unless indicated otherwise.

For $n \geq 2$, let P_n denote a path on n vertices, and for $n \geq 3$, let C_n denote a cycle on n vertices, where $V(P_n) = V(C_n) = \{0, \ldots, n-1\}$, and where adjacencies are defined in the natural way.

The Kronecker product $G \times H$ of graphs $G = (V, E)$ and $H = (W, F)$ is defined as follows: $V(G \times H) = V \times W$ and $E(G \times H) = \{\{(a, x), (b, y)\} : \{a, b\} \in E \text{ and } \{x, y\} \in F\}$. (This product is variously known as direct product, tensor product, cardinal product, cross product and graph conjunction.) Further, the *Cartesian product* $G \Box H$ of graphs G and H defined as follows: $V(G \Box H) = V \times W$ and $E(G \Box H) = \{\{(a, x), (b, y)\} : \{a, b\} \in E \text{ and } x = y, \text{ or } \{x, y\} \in F \text{ and } a = b\}$. Unfortunately, there is no unanimity on the notation. For example, $G \Box H$ also appears as $G \times H$ in the literature.

Each of the foregoing product operations is commutative and associative up to isomorphism. The graph $P_m \Box P_n$ is known as an $m \times n$ grid. It has mn vertices and $m(n-1) + (m-1)n$ edges. The following are certain salient characteristics of $C_m \times C_n$ [15]:

1. $C_m \times C_n$ is a non-planar graph.
2. $C_m \times C_n$ is bipartite if and only if m or n is even.
3. If m or n is odd, then $C_m \times C_n$ is a connected graph, and if m and n are both even, then $C_m \times C_n$ consists of two isomorphic components.
4. Each component of $C_m \times C_n$ admits an edge decomposition into two Hamiltonian cycles.

For any undefined terms, see [15].

The Kronecker product is challenging to deal with

Among the four standard graph products, (viz., Cartesian product, Kronecker product, strong product and lexicographic product), the one that is most difficult to deal with is the Kronecker product. Here are some supporting arguments:

1. A product of two connected graphs relative to each of the other three operations is necessarily connected – a fact that is easy to prove. On the other hand, the Kronecker product of two connected graphs need not be connected – a fact not obvious at all.
2. A graph G is necessarily a subgraph of its product with a nonempty graph as far as the other three operations are concerned. However, the analogous statement with respect to the Kronecker product is far from true. For example, if m is odd and n is even, then C_m cannot appear as a subgraph of $C_m \times C_n$ for the simple reason that $C_m \times C_n$ in this case is bipartite. Worse, graphs G exist such that G is non-planar, yet $G \times K_2$ is planar [2].
3. Whereas the distance between two vertices in a product graph with respect to each of the other three operations is given by a simple formula [15], that with respect to the $\times$-product is given by a formula that is unusually complicated [21].

Challenges notwithstanding, there are many graphs built around this product that are amenable to applications in engineering and computer science. As stated earlier, if m and n are both odd, then $C_m \times C_n$ outperforms $C_m \Box C_n$ in many ways. Further, $C_m \times C_n$ has a rich cycle structure [18].

State of the art

Of all graph products, the Cartesian product has received maximum attention in the literature. This is mainly because this product is intuitive; in particular, it inherits the factor graphs in an obvious way.

It is easy to see that $\mathrm{cr}(C_m \Box C_n) \leq (m-2)n$ where $m \leq n$. Harary et al. [13] conjectured in 1973 that the inequality in the preceding statement is an equality. Indeed, investigations in this direction suggest that this is probably true. To that end, Ringeisen and Beineke [29,3] showed that $\mathrm{cr}(C_m \Box C_n) = (m-2)n$ for $m = 3, 4$ as Dean and Richter [6] later provided the missing details about $C_4 \Box C_4$. The largest value of m for which the foregoing conjecture has been verified is 7 [1]. In a major advance, Glebsky and Salazar [12] proved in 2004 that the conjecture holds for $n \geq m(m+1)$.

In a related development, Klešč [22] obtained exact values of the crossing numbers of the $\Box$-products of cycles with four special graphs of order five. He [23] subsequently examined the analogous problem with respect to the join of certain special graphs. Circulant graphs and generalized Petersen graphs have also been studied in this direction [25,30].

What follows

Section 2 consists of certain lower bounds on $\mathrm{cr}(C_m \times C_n)$. We take an indirect approach and show that $C_m \times C_n$ contains $C_m \Box C_{\lfloor n/2 \rfloor}$ as a minor, and utilize the existing results to develop the lower bounds. Sections 3 and 4 deal with the orthogonal drawings of $C_m \times C_n$ for (1) m odd and n even, and (2) m and n both odd, respectively. Section 5 treats the special case of m odd and n a multiple of m. The resulting upper bounds are a lot more impressive. Finally Section 6 summarizes the results and presents certain concluding remarks. (We do not address the case when m and n are both even, since $C_m \times C_n$ in that case consists of two connected components isomorphic to each other, and it turns out that each such component is similar to $C_m \times C_n$ where m is odd and n is even [20].)

In the rest of the paper, the arithmetic on vertices in $C_m \times C_n$ is modulo m in the first co-ordinate and modulo n in the second co-ordinate.

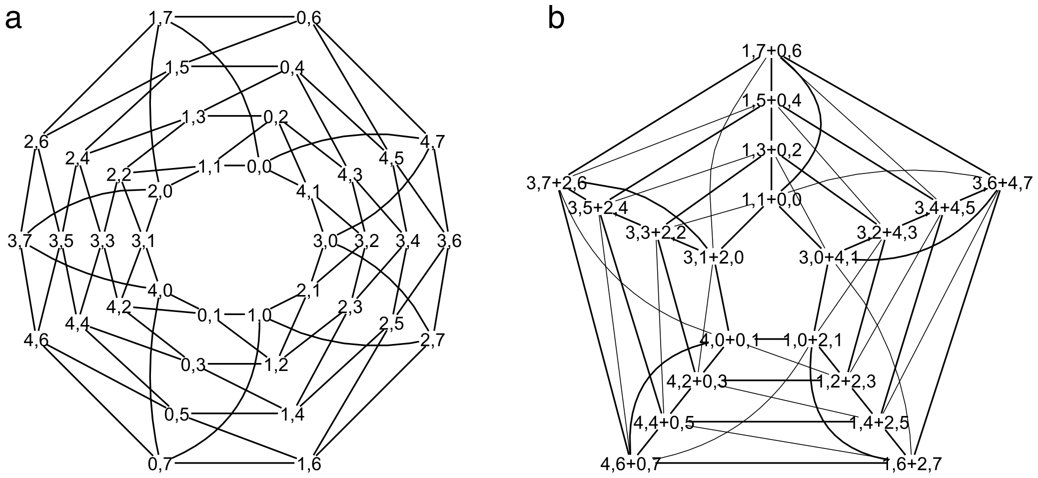

Fig. 1. (a) $C_5 \times C_8$ and (b) a minor of $C_5 \times C_8$.

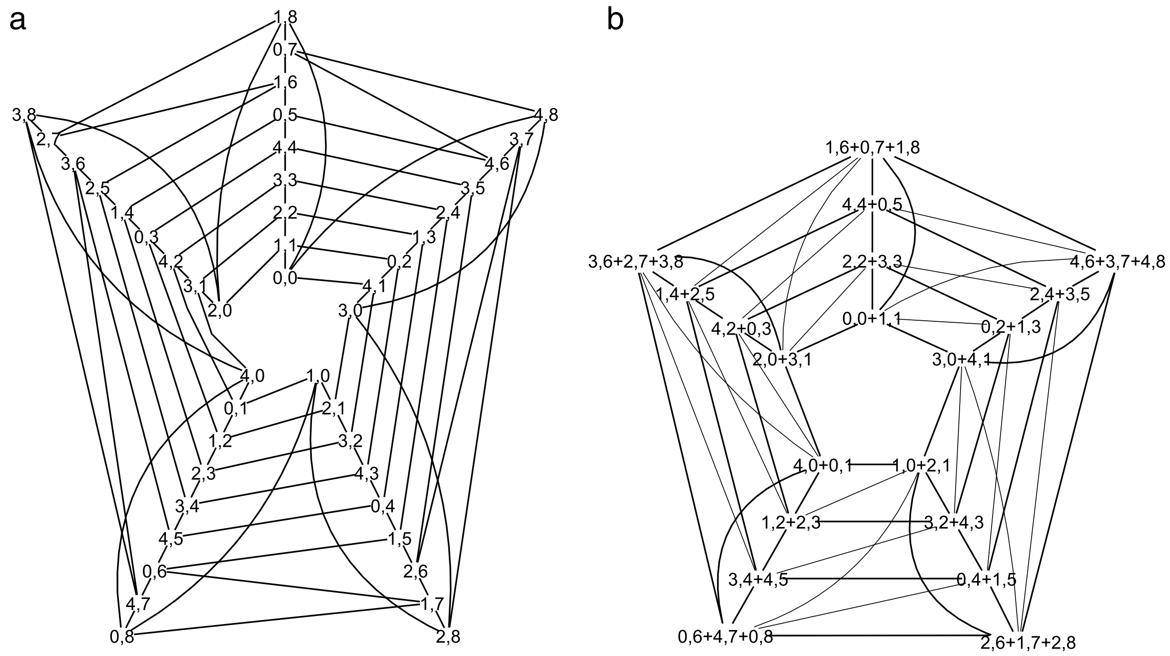

Fig. 2. (a) $C_5 \times C_9$ and (b) a minor of $C_5 \times C_9$.

2. Lower bounds on cr($C_m \times C_n$)

Obtaining a nontrivial lower bound on the crossing number of a graph is known to be a very difficult task. The situation is no different in the present study.

Our method of attack is as follows: (1) show that $C_m \times C_n$ contains $C_m \square C_{\lfloor n/2 \rfloor}$ as a minor, (2) invoke an existing connection between the crossing number of a graph and that of its minor, and (3) utilize the known lower bounds on cr($C_m \square C_n$). Note that the binary relation "is a minor of" is transitive.

Lemma 2.1. *If m is odd and n is even, where $n \geq 6$, then $C_m \times C_n$ contains $C_m \square C_{n/2}$ as a minor.*

Proof. For m odd and n even, $C_m \times C_n$ may be viewed as a graph containing $n/2$ "concentric" cycles, each of length $2m$. See Fig. 1(a) in respect of $m = 5$ and $n = 8$. A careful contraction of m alternate edges (appearing in the same relative position) in each of these cycles leads to a six-regular graph on $mn/2$ vertices. See Fig. 1(b) for an illustration. It is easy to see that the resulting graph includes the four-regular $C_m \square C_{n/2}$ as a subgraph. □

Lemma 2.2. *If m and n are both odd, where $m \leq n$ and $n \geq 7$, then $C_m \times C_n$ contains $C_m \square C_{(n-1)/2}$ as a minor.*

Proof. The graph $C_m \times C_n$ admits a vertex partition into m (shortest odd) cycles, each of length n [19]. The following is an outline of the proof.

Let σ_0 denote the sequence $(a_0, 0), (a_1, 1), \ldots, (a_{n-1}, n-1)$, where $a_i = i$ for $0 \leq i \leq m-1$, and $a_i = (i+1) \bmod 2$ for $m \leq i \leq n-1$. It is easy to see that σ_0 constitutes a cycle of length n in $C_m \times C_n$. For $1 \leq i \leq m-1$, consider the sequence σ_i given by $(a_0 + i, 0), \ldots, (a_{n-1} + i, n-1)$, where the sum $a_j + i$ is modulo m. Check to see that $\sigma_0, \sigma_1, \ldots, \sigma_{m-1}$ constitute a vertex partition of $C_m \times C_n$ into m cycles, each of length n. (The remaining mn edges constitute an analogous partition.) See Fig. 2(a) in respect of $m = 5$ and $n = 9$ [36].

A careful contraction of $(n+1)/2$ edges (appearing in the same relative position) in each σ_i leads to a six-regular graph on $m(n-1)/2$ vertices. See Fig. 2(b) for an illustration. It is easy to see that the resulting graph includes the four-regular $C_m \square C_{(n-1)/2}$ as a subgraph. □

Towards the desired bounds, we invoke the technical results in the following theorem.

Theorem 2.3. 1. [26]. *$C_m \times C_n$ and $C_m \Box C_n$ are isomorphic to each other if and only if m and n are both odd and equal.*
2. [10]. *If G is a graph and M is a minor of G such that the maximum degree of M is at most four, then* $\mathrm{cr}(G) \geq \frac{1}{4}\mathrm{cr}(M)$.
3. [31]. *For each $\epsilon > 0$, there exists a (sufficiently large) integer n_0 such that* $\mathrm{cr}(C_m \Box C_n) \geq (0.8 - \epsilon)mn$ *for* $n \geq m \geq n_0$. □

The following result is immediate.

Corollary 2.4. $\mathrm{cr}(C_m \times C_n)$ *is greater than or equal to*

$$\begin{cases} (0.8-\epsilon)mn & m, n \text{ odd and equal}, m \geq n_0 \\ \frac{1}{8}(0.8-\epsilon)mn & m \text{ odd}, n \text{ even}, n \geq 6, \min\{m, n/2\} \geq n_0 \\ \frac{1}{8}(0.8-\epsilon)m(n-1) & \\ \quad m, n \text{ odd}, m < n, n \geq 7, \min\{m, (n-1)/2\} \geq n_0 \end{cases}$$

where, in each case, $\epsilon > 0$ and n_0 is a sufficiently large integer depending only on ϵ. □

In certain cases, we get a slightly better lower bound by using the following (groundbreaking) result [12] in place of Theorem 2.3(3): $\mathrm{cr}(C_m \Box C_n) = (m-2)n$ for $n \geq m(m+1)$.

For the special case of $n = 3, 5, 7$, we get the exact $\mathrm{cr}(C_n \times C_n) = (n-2)n$ that is based on Theorem 2.3(1) and the fact that $\mathrm{cr}(C_n \Box C_n) = (n-2)n$ if $n = 3, 5, 7$ [1].

3. Product of an odd cycle and an even cycle

Throughout this section, m is odd and n is even. We first present a vertex partition of $C_m \times C_n$ into two isomorphic grids.

Lemma 3.1. *If m is odd and n is even, then $C_m \times C_n$ admits a vertex partition into two grids isomorphic to $P_{n/2} \Box P_m$ and $P_m \Box P_{n/2}$, respectively.*

Proof. Let σ_0 denote the sequence $(0, a_0), (1, a_1), \ldots, (m-1, a_{m-1})$, where $a_0 = n-1, a_1 = 0$ and $a_i = i-1$, where $2 \leq i \leq m-1$. Further, consider the sequence σ_j given by $(j, a_0 - j), (j+1, a_1 - j), \ldots, (j+m-1, a_{m-1} - j)$, where $1 \leq j \leq n/2 - 1$. Check to see that the sequences $\sigma_0, \ldots, \sigma_{n/2-1}$ are mutually vertex-disjoint, and they collectively correspond to a grid isomorphic to $P_{n/2} \Box P_m$.

Next, let μ_0 denote the sequence $(b_0, 0), (b_1, 1), \ldots, (b_{n/2-1}, n/2-1)$, where $b_0 = m-1, b_1 = 0$ and $b_i = i-1$, where $2 \leq i \leq n/2 - 1$. Further, consider the sequence μ_j given by $(b_0 - j, j), (b_1 - j, j+1), \ldots, (b_{n/2-1} - j, j + n/2 - 1)$, where $1 \leq j \leq m-1$. Check to see that the sequences $\mu_0, \ldots, \mu_{m-1}$ are mutually vertex-disjoint, and they collectively correspond to a grid isomorphic to $P_m \Box P_{n/2}$. Further, the two grids thus constructed constitute a vertex partition of $C_m \times C_n$. □

An illustration for the proof of Lemma 3.1 appears in Fig. 3 in respect of $C_5 \times C_6$. The following is our algorithm for an orthogonal embedding of $C_m \times C_n$ for $m \geq n/2$. (The other case is similar.)

Algorithm A.

Step 1: Embed the twin grids from the proof of Lemma 3.1 such that the top row of the left grid is horizontally aligned with that of the right grid as in Fig. 3. The cumulative number of edges in the two grids is given by $2(m(n/2-1) + (m-1)n/2)$ that is equal to $2mn - (2m+n)$ in $C_m \times C_n$ that has a total of $2mn$ edges.

Step 2: There are $m - n/2$ edges that appear as a matching in the left grid between the rightmost $m - n/2$ vertices in its top row and the leftmost $m - n/2$ vertices in its bottom row. Further, there exists an identical matching in the right grid. Introduce the corresponding edges as in Fig. 4. Each edge in these matchings renders $(n-3)$ crossings, hence the number of edge crossings at this step is equal to $2(m - n/2)(n-3)$.

Step 3: The remaining $2n$ edges run between the two grids as four disjoint matchings of $n/2$ edges each. See Fig. 5. The number of edge crossings rendered by each of these matchings is given by $1 + 2 + \cdots + (n/2 - 1)$ that is equal to $\frac{1}{2}(n/2-1)n/2$, hence the number of edge crossings introduced at this step is equal to $n(n/2-1)$. □

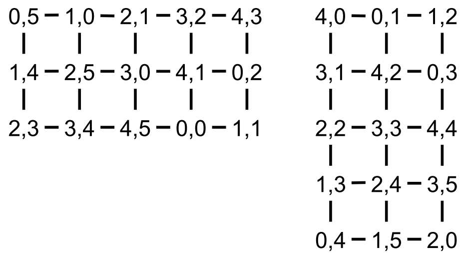

Fig. 3. Vertex partition of $C_5 \times C_6$ into two grids.

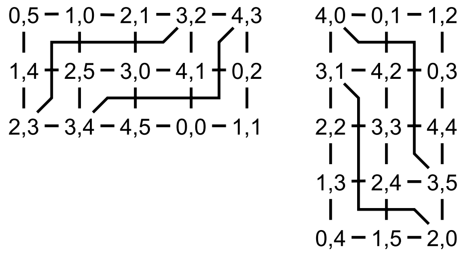

Fig. 4. Step 2 of Algorithm A in respect of $C_5 \times C_6$.

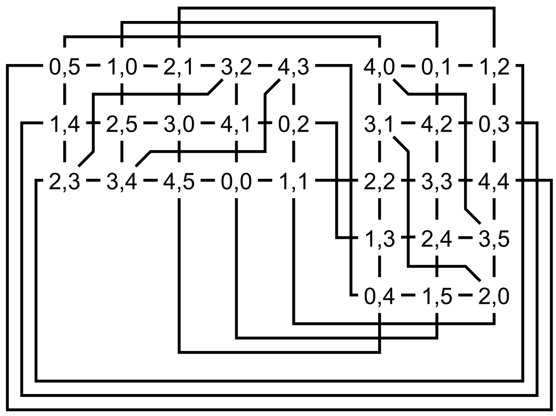

Fig. 5. Final step of Algorithm A in respect of $C_5 \times C_6$.

For $m \geq n/2$, Algorithm A leads to $\mathrm{cr}(C_m \times C_n) \leq 2(m - n/2)(n-3) + n(n/2-1)$. For $m < n/2$, the embedding is obtainable in an analogous fashion. See Fig. 6 that illustrates this case in respect of $C_5 \times C_{14}$. The following result is immediate.

Theorem 3.2. *If m is odd and n is even, where $m \geq 3$ and $n \geq 4$, then $\mathrm{cr}(C_m \times C_n)$ is less than or equal to*

$$\begin{cases} 2(m-n/2)(n-3) + n(n/2-1), & \text{if } m \geq n/2 \\ 2(n/2-m)(2m-3) + 2m(m-1), & \text{if } m < n/2. \end{cases}$$ □

Remark. Embedding of some of the edges in Figs. 5 and 6 are not orthogonal. However, each such edge may easily be embedded in an orthogonal fashion by introducing a couple of additional bends. The number of edge crossings stays the same.

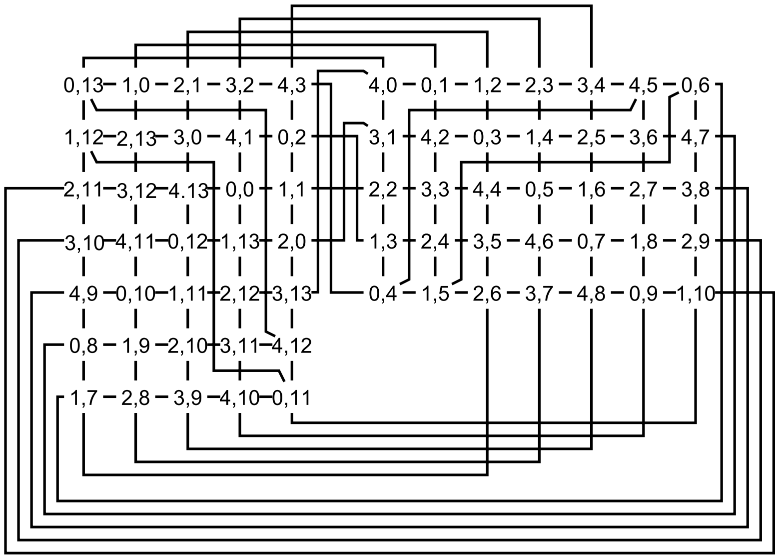

Fig. 6. Orthogonal embedding of $C_5 \times C_{14}$.

Assume that m is an arbitrary but fixed positive odd integer. For $4 \le n \le 2m$, the upper bound from Theorem 3.2 may be written as $-n^2/2 + 2(m+1)n - 6m$ that is a negative quadratic in n, so it grows slowly. For $n > 2m$, the upper bound is linear in n. This observation is depicted in Fig. 17 in Section 6 in respect of $C_{45} \times C_n$.

4. Product of two odd cycles

Unlike the product of an odd cycle and an even cycle, the product of two odd cycles is challenging to deal with. This is probably because the former is bipartite while the latter is non-bipartite.

Throughout this section, m and n are odd integers greater than or equal to three, and $m < n$. We begin with a partition of $C_m \times C_n$ into two (non-isomorphic) grids.

Lemma 4.1. *If m and n are both odd and $m < n$, then $C_m \times C_n$ admits a vertex partition into two grids isomorphic to $P_{(n+m)/2} \square P_m$ and $P_m \square P_{(n-m)/2}$, respectively.*

Proof. Let σ_0 denote the sequence $(0, a_0), (1, a_1), \ldots, (m-1, a_{m-1})$, where $a_0 = n-(m+1)$, $a_1 = n-m, \ldots, a_{m-1} = n-2$. Further, consider the sequence σ_j given by $(j, a_0 - j), (j+1, a_1 - j), \ldots, (j+m-1, a_{m-1}-j)$, where $1 \le j \le (n+m)/2-1$. Check to see that $\sigma_0, \ldots, \sigma_{(n+m)/2-1}$ are mutually vertex-disjoint and they collectively correspond to a grid isomorphic to $P_{(n+m)/2} \square P_m$.

Next, let μ_0 be the sequence $(0, b_0), (1, b_1), \ldots, ((n-m)/2-1, b_{(n-m)/2-1})$, where $b_0 = n-1$, $b_1 = 0$ and $b_i = i-1$, where $2 \le i \le (n-m)/2-1$. Further, consider the sequence μ_j given by $(j, b_0-j), (j+1, b_1-j), \ldots, (j+(n-m)/2-1, b_{(n-m)/2-1}-j)$, where $1 \le j \le m-1$. Check to see that $\mu_0, \ldots, \mu_{m-1}$ are mutually vertex-disjoint and they collectively correspond to a grid isomorphic to $P_m \square P_{(n-m)/2}$. Further, the two grids constitute a vertex partition of $C_m \times C_n$. □

Fig. 7 illustrates the proof of Lemma 4.1 in respect of $C_5 \times C_{11}$. Towards an algorithm for an orthogonal embedding of $C_m \times C_n$, we distinguish between two cases: (1) $m < n < 3m$, i.e., $(n-m)/2 < m$, and (2) $n \ge 3m$, i.e., $(n-m)/2 \ge m$. Here is the scheme for the first case.

Algorithm B.

Step 1: Embed the two grids from the proof of Lemma 4.1 such that the top row of the (larger) left grid is horizontally aligned with that of the (smaller) right grid as in Fig. 7. The cumulative number of edges in the two grids is given by $((n+m)/2-1)m + (m-1)(n+m)/2 + (m-1)(n-m)/2 + m((n-m)/2-1)$ that is equal to $2mn - (2m+n)$ in the graph that has $2mn$ edges.

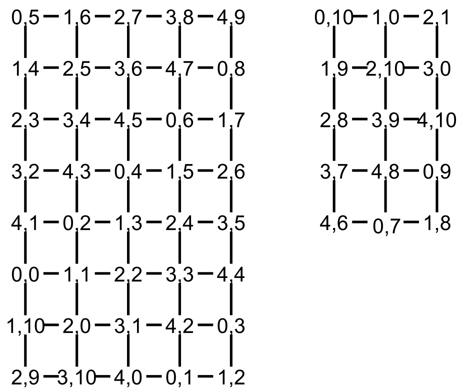

Fig. 7. Proof of Lemma 4.1 in respect of $C_5 \times C_{11}$.

Step 2: There are m edges that appear as a matching between the top m vertices in the rightmost column of the left grid and the m vertices in the leftmost column of the right grid. Introduce these edges. Further, there exists a matching between the bottom m vertices in the leftmost column of the left grid and the m vertices in the rightmost column of the right grid. Introduce the corresponding m edges. See Fig. 8 for an illustration.

Step 3: There are $(n-m)/2$ edges that appear as a matching in the left grid between the top $(n-m)/2$ vertices in its leftmost column and the bottom $(n-m)/2$ vertices in its rightmost column. Further, there exists a matching between the $(n-m)/2$ vertices in the bottom row of the right grid and the leftmost $(n-m)/2$ vertices in the top row of the left grid. Introduce the $(n-m)$ edges corresponding to these matchings as in Fig. 9. The number of edge crossings at this step is equal to $m(n-m)$.

Step 4: The remaining m edges appear as a matching between two sets, the first of which consists of the m vertices in the bottom row of the left grid while the second consists of the rightmost $m-(n-m)/2$ vertices in the top row of the left grid and all $(n-m)/2$ vertices in the top row of the second grid. See Fig. 10. Each edge in the present matching renders $(n-3)$ crossings, hence the number of edge crossings at this step is equal to $m(n-3)$. □

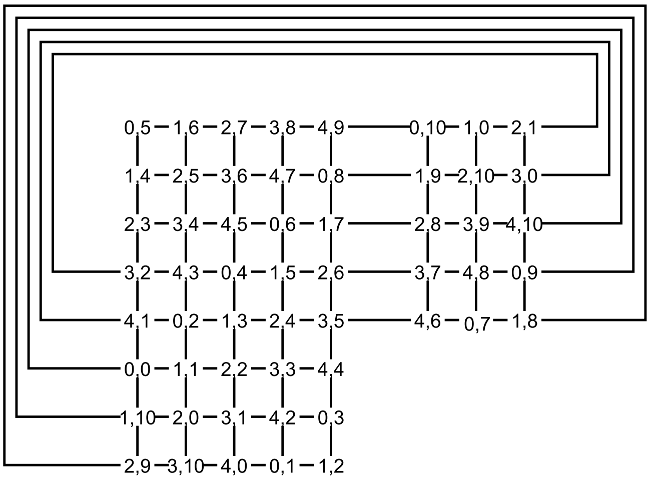

Fig. 8. Step 2 of Algorithm B in respect of $C_5 \times C_{11}$.

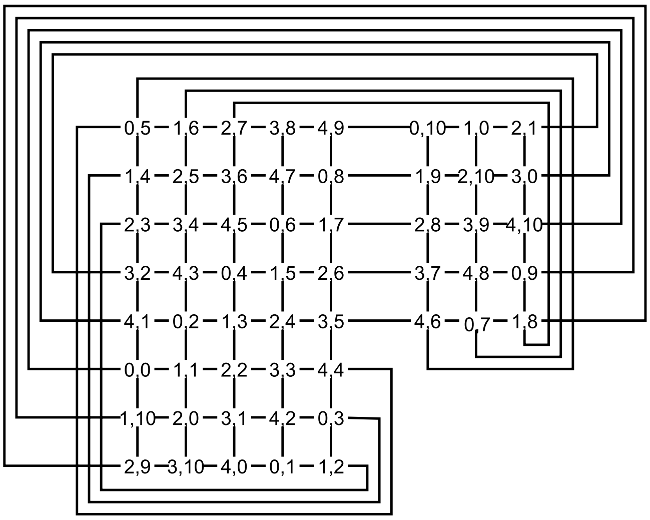

Fig. 9. Step 3 of Algorithm B in respect of $C_5 \times C_{11}$.

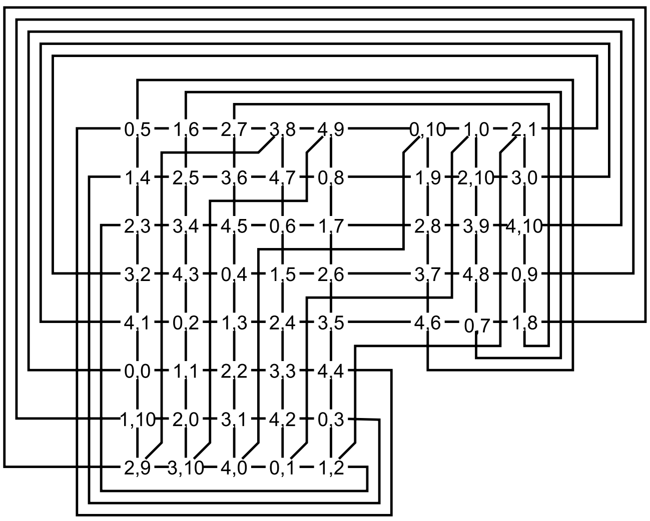

Fig. 10. Final step of Algorithm B in respect of $C_5 \times C_{11}$.

The following result is immediate:

Theorem 4.2. *If m and n are both odd and $m < n < 3m$, then* $\mathrm{cr}(C_m \times C_n) \le 2mn - (m^2 + 3m)$. □

In the remainder of this section, let n be greater than or equal to $3m$. Here is the algorithm for this case.

Algorithm B′.

Steps 1 through 2: Same as those in Algorithm B.

Step 3: There exists a matching in the left grid between the top $(n-m)/2$ vertices in its leftmost column and the bottom $(n-m)/2$ vertices in its rightmost column. Further, there exists another matching of the same size in which (a) the leftmost m vertices in the bottom row of the right grid are connected to the m vertices in the top row of the left grid, and (b) the rightmost $(n-m)/2 - m$ vertices in the bottom row of the right grid are connected to as many leftmost vertices in the top row of the same grid. Introduce the $(n-m)$ edges corresponding to these matchings. The total number of edges in the embedding thus far is equal to $2mn - m$ and the number of edge crossings at this step is equal to $(2m-3)(n-m)/2 + m^2 + (2m-3)((n-m)/2 - m)$ that is equal to $(2mn + 6m) - (3m^2 + 3n)$.

Step 4: The remaining m edges appear as a matching between the m vertices in the bottom row of the left grid and the m rightmost vertices in the top row of the right grid. See Fig. 11. The number of edge crossings at this step is equal to m^2. □

Theorem 4.3. *If m and n are both odd and $n \ge 3m$, then* $\mathrm{cr}(C_m \times C_n) \le (2mn + 6m) - (2m^2 + 3n)$. □

Assume that m is an arbitrary but fixed positive odd integer. By Theorem 4.2, the upper bound for $C_m \times C_{3m-2}$ is equal to $5m^2 - 7m$. Further, by Theorem 4.3, the upper bound for $C_m \times C_{3m+2}$ is equal to $4m^2 + m - 6$ that is smaller than $5m^2 - 7m$ if $m \ge 9$. This observation is depicted in Fig. 18 in Section 6 in a more general setting. We suspect that $\mathrm{cr}(C_m \times C_n)$ has a similar drop around $n = 3m$.

5. A special case

It turns out that if n is an integer multiple of m, then the drawings and the resulting upper bounds on $\mathrm{cr}(C_m \times C_n)$ are a lot better than in Sections 3 and 4.

Lemma 5.1. *If m is odd and n is a multiple of m, then $P_m \square P_n$ appears as a spanning subgraph of $C_m \times C_n$.*

Proof. Let σ_0 denote the sequence $(a_0, b_0), (a_1, b_1), \ldots, (a_{n-1}, b_{n-1})$, where $a_i = i \bmod m$ and $b_i = i$, $0 \le i \le n-1$. Further, consider the sequence σ_j given by $(a_0 + j, b_0 - j), (a_1 + j, b_1 - j), \ldots, (a_{n-1} + j, b_{n-1} - j)$, $1 \le j \le m-1$. Check to see that the sequences $\sigma_0, \ldots, \sigma_{m-1}$ are mutually vertex-disjoint, and they collectively correspond to a grid isomorphic to $P_m \square P_n$. □

The proof of Lemma 5.1 is illustrated in Fig. 12 in respect of $C_5 \times C_{20}$.

Note: It is easy to see that the spanning grid $P_m \square P_n$ from Lemma 5.1 may be extended to the spanning "prism" $P_m \square C_n$, which means that under the conditions of Lemma 5.1, $P_m \square C_n$ is a (large) common subgraph of $C_m \times C_n$ and $C_m \square C_n$. This is interesting, since $C_m \times C_n$ and $C_m \square C_n$ are known to be non-isomorphic with the sole exception of when m and n are both odd and equal.

The following is our algorithm for an orthogonal drawing of $C_m \times C_n$, where m is odd and $n = km$, $k \ge 2$. First suppose that k is even.

Algorithm C.

Step 1: Consider the spanning grid $P_m \square P_n$ of $C_m \times C_n$ from the proof of Lemma 5.1. Remove the necessary edges from the grid to obtain k "square" (sub)grids, each isomorphic to $P_m \square P_m$, and embed them in the plane as shown in Fig. 13 in respect of $m = 5$ and $n = 20$. (Indexing of the grids is not a part of the drawing.) The number of edges in this collection is equal to $2(m-1)n$.

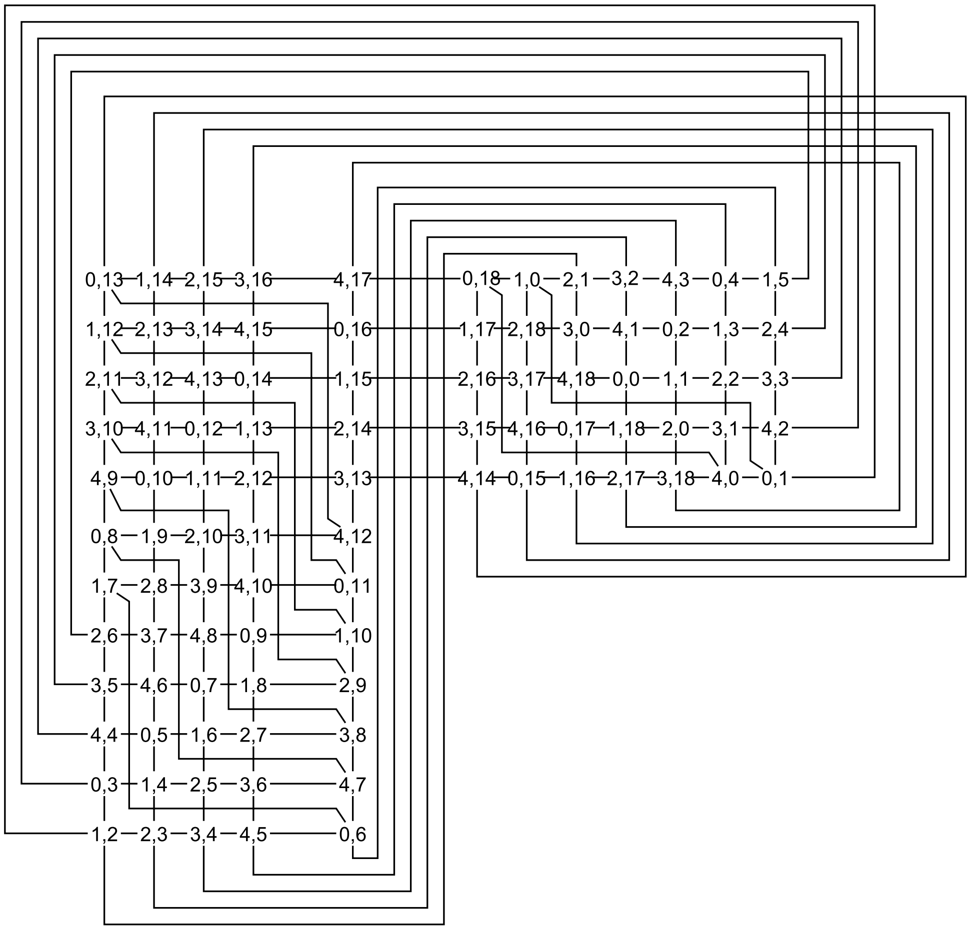

Fig. 11. Algorithm B′ in respect of $C_5 \times C_{19}$.

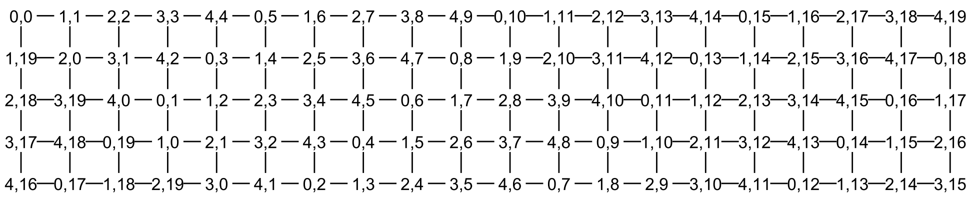

Fig. 12. Proof of Lemma 5.1 in respect of $C_5 \times C_{20}$.

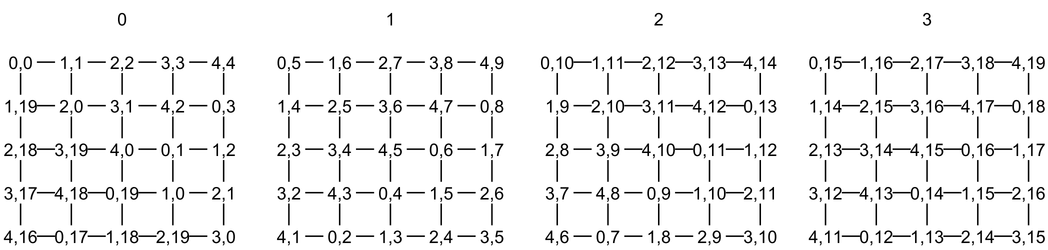

Fig. 13. Step 1 of Algorithm C in respect of $C_5 \times C_{20}$.

Remark. Edges of $C_m \times C_n$ not yet in the foregoing collection run between two subgrids if and only if the corresponding indices differ by one or $k - 1$.

Step 2: Mirror each of the odd-indexed subgrids on the x-axis. See Fig. 14 in respect of the running example.

Step 3: For each i, $0 \le i \le k - 1$, there exist $2m$ edges between the ith subgrid and the $(i + 1)$st subgrid, and they appear as two disjoint matchings of m edges each. Introduce all such edges as in Fig. 15. The number of crossings in each matching is equal to $1+2+\cdots+(m-1) = (m-1)m/2$. It is easy to see that the total number of crossings is equal to $2k(m-1)m/2 = (m-1)n$. □

Theorem 5.2. *If m is odd and $n = km$ where $k \ge 2$ and k is even, then* $\mathrm{cr}(C_m \times C_n) \le (m-1)n$. □

For the case when k is odd, the only change occurs in Step 3 in which the number of crossings among edges running between the $(k-1)$th subgrid and the 0th subgrid is equal to m^2 instead of $m(m-1)$. See Fig. 16 in respect of $C_5 \times C_{25}$.

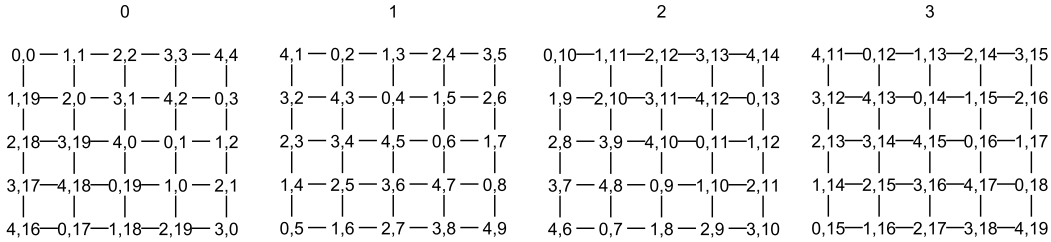

Fig. 14. Step 2 of Algorithm C in respect of $C_5 \times C_{20}$.

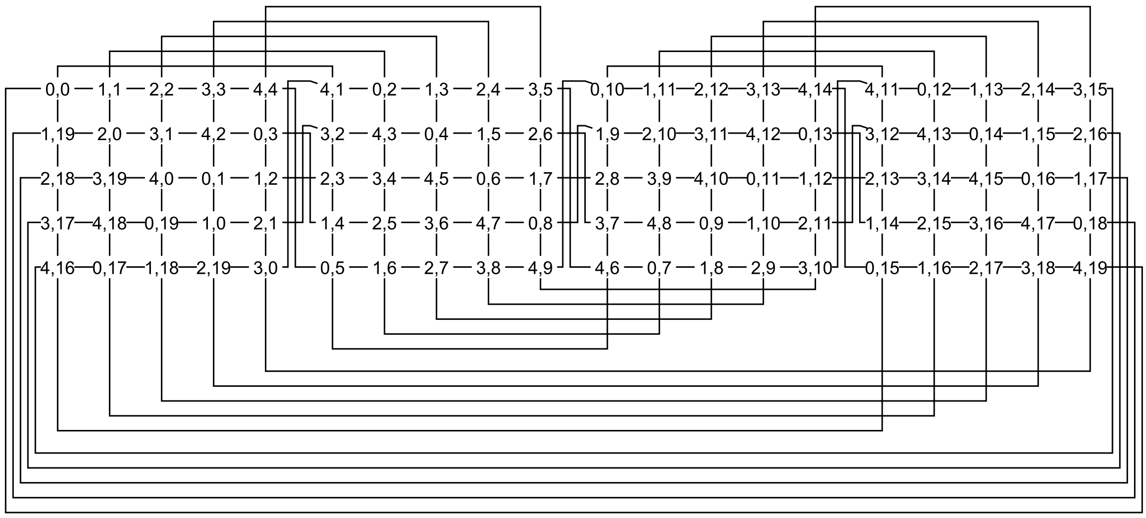

Fig. 15. Final step of Algorithm C in respect of $C_5 \times C_{20}$.

Theorem 5.3. *If m is odd and $n = km$ where $k \geq 3$ and k is odd, then* $\mathrm{cr}(C_m \times C_n) \leq (m-1)n + m$. □

6. Concluding remarks

We present orthogonal drawings of $C_m \times C_n$ with an express objective of achieving good bounds on its crossing number. There are two main cases: (1) m odd and n even, and (2) m and n both odd; in addition, there is a special case of n being a multiple of m. It is easy to see that the embedding area is $O((m+n)^2)$ in the first two cases and $O(mn)$ in the third. Obtaining a drawing with the minimum embedding area is known to be NP-hard [11].

All drawings are built around a vertex partition of $C_m \times C_n$ into grids that may be viewed as clusters. In each case, the number of edges of the graph not in the underlying grids is $O(m+n)$. Further, the maximum number of bends in each such edge is a constant, hence the total number of bends in the drawings is also $O(m+n)$. Obtaining an orthogonal drawing with the fewest bends is known to be NP-hard [7].

Our schemes are linear in $|E(C_m \times C_n)|$. The following is a summary of the upper bound, say $\overline{\mathrm{cr}}(C_m \times C_n)$, on $\mathrm{cr}(C_m \times C_n)$, where m is odd:

$$\begin{cases} (m-2)n & n = m \\ (m-1)n & n = km,\ k \text{ even} \geq 2 \\ (m-1)n + m & n = km,\ k \text{ odd} \geq 3 \\ (2m-n)(n-3) + n(n/2-1) & n \text{ even},\ m \geq n/2 \\ (n-2m)(2m-3) + 2m(m-1) & n \text{ even},\ n/2 > m \\ 2mn - (m^2 + 3m) & n \text{ odd},\ m < n < 3m \\ (2mn + 6m) - (2m^2 + 3n) & n \text{ odd},\ n \geq 3m. \end{cases}$$

(Take the minimum as appropriate.)

It is easy to see that the foregoing upper bound is within a constant multiple of the lower bound from Corollary 2.4. Let $\varepsilon(C_m \times C_n)$ denote the ratio of $\overline{\mathrm{cr}}(C_m \times C_n)$ to $|E(C_m \times C_n)|$. It is clear that this normalized quantity is less than one. Further, if m is assumed to be fixed and n is even with $m \geq n/2 \geq 2$, then it is given by $(-1/(4m))n + (m+1)/m - 3/n$ that reaches its maximum at $n = \sqrt{12m}$. Also, if n is an integer multiple of m, then it is equal to nearly one-half. In case the preceding conditions are not true and n is sufficiently large, then $\varepsilon(C_m \times C_n)$ is given by

$$\begin{cases} 1 - (3/(2m) + (m-2)/n) & n \text{ even and } n \geq 2m \\ 1 - (3/(2m) + (m-3)/n) & n \text{ odd and } n \geq 3m. \end{cases}$$

Fig. 17 traces $\overline{\mathrm{cr}}(C_{45} \times C_n)$ and $\varepsilon(C_{45} \times C_n)$ for even n while Fig. 18 does that for odd n, where $3 \leq n \leq 274$. The following are certain observations on the bounds on $\mathrm{cr}(C_m \times C_n)$:

- The results are most impressive when m and n are both odd and equal.
- The next best case arises when m is odd and $n = km$, where $k \geq 2$. Indeed, as k goes up, the width of the drawing gets progressively larger than its height.
- The worst case arises when m and n are wide apart and the foregoing relationship does not exist between m and n.

Bridging the gap between the bounds on $\mathrm{cr}(C_m \times C_n)$ seems to be an interesting problem.

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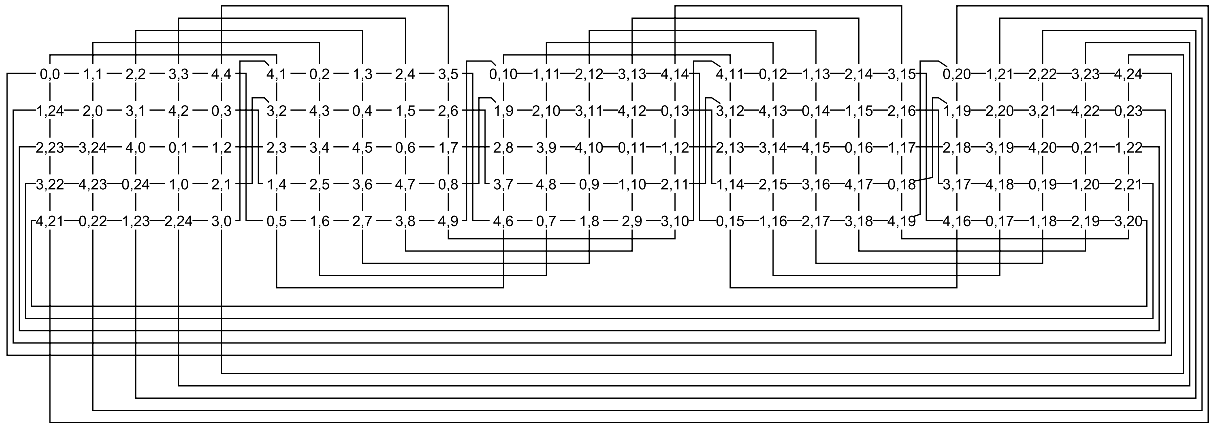

Fig. 16. Drawing of $C_5 \times C_{25}$.

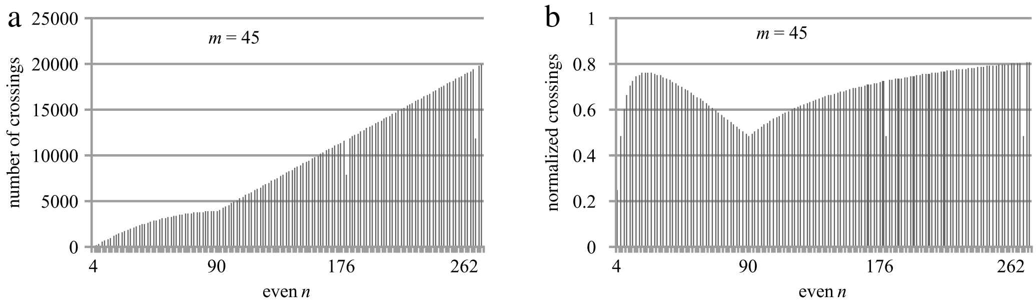

Fig. 17. (a) $\overline{\mathrm{cr}}(C_{45} \times C_n)$ and (b) $\varepsilon(C_{45} \times C_n)$, vs. even n, $4 \leq n \leq 274$.

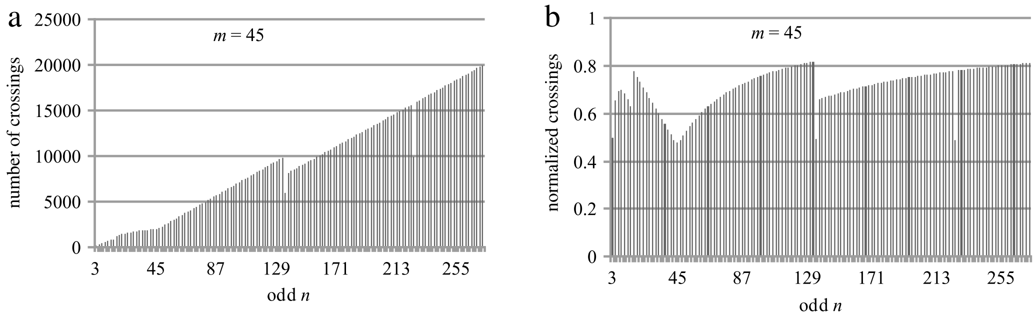

Fig. 18. (a) $\overline{\mathrm{cr}}(C_{45} \times C_n)$ and (b) $\varepsilon(C_{45} \times C_n)$, vs. odd n, $3 \leq n \leq 273$.

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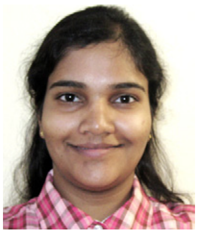
Dr. Jha is a member of Upsilon Pi Epsilon.



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