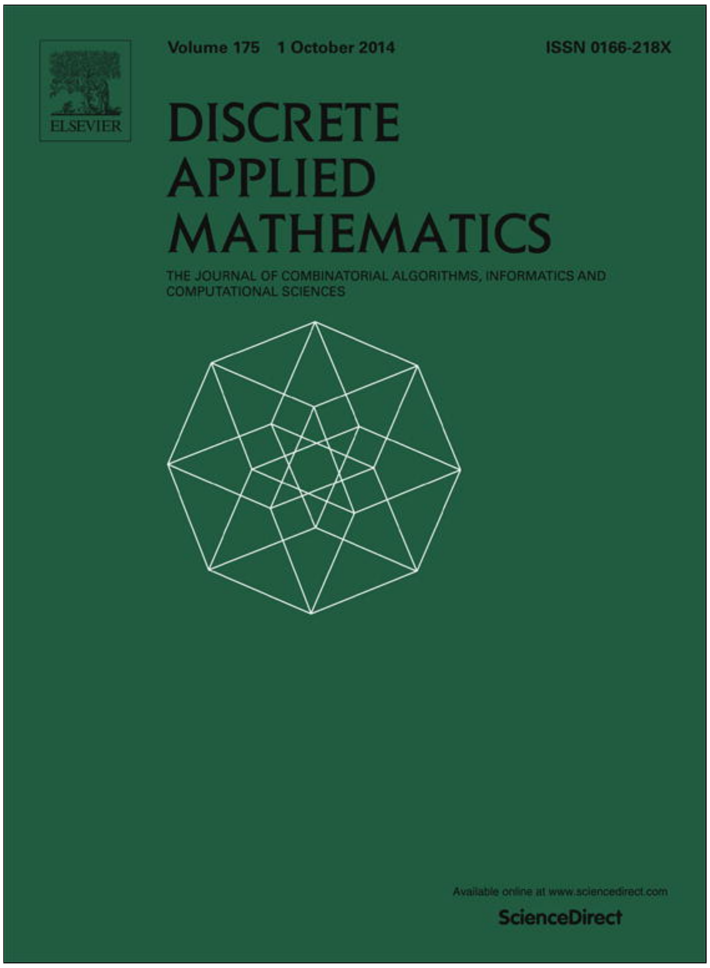
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Tight-optimal circulants vis-à-vis twisted tori

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ABSTRACT

In 1991, Tzvieli presented several families of optimal four-regular circulants. Prominent among them are three families that include graphs having $(2a + d)a$ vertices for each $a \geq 5$, where $d = -1, 0, +1$. The step sizes in each case are 1 and $(2a + d)k - 1$, where $\gcd(a, k) = 1$ and $1 \leq k \leq \lfloor \frac{1}{2}(a - 1) \rfloor$. For $d = 0$, the graphs are called dense bipartite circulants, which were studied at length by the author recently. This paper examines the other two families and shows that the circulants in each of them are systematically obtainable from the twisted torus $TT(2a + d, a)$ by trading up to $2a$ edges for as many new edges, where $d = -1, +1$. In the process, the graphs seamlessly inherit all good characteristics of the twisted torus. In particular, each circulant in each family is tight-optimal, hence its average distance is the least among all circulants of the same order and size. Further, it admits a perfect dominating set under certain conditions on a and k.

1. Introduction

The *circulant graphs*, which we formally define below, possess several excellent features that render them fit for an application as a *network topology* in areas such as parallel/distributed systems and VLSI [1–3,12]. In a deep study, Tzvieli [13] earlier presented several families of *four-regular optimal circulants*. The graphs in three such families are as follows:

$$\mathcal{C}_{(2a+d)a}\big(1,\,(2a+d)k-1\big)\Big|_{d=-1,0,+1}\; a \geq 5;\quad 1 \leq k \leq \left\lfloor \frac{1}{2}(a-1) \right\rfloor;\ \text{and}\ \gcd(a,k) = 1.$$

The *diameter* of each of the foregoing graphs is equal to a. Some other characteristics appear in Table 1. (The implicit claims will be proved later.)

Among the three kinds in Table 1, $\mathcal{C}_{2a^2}(1, 2ak - 1)$ was studied at length recently [8], so the focus in this paper is on the other two. We show that $\mathcal{C}_{(2a+d)a}(1, (2a + d)k - 1)$ is obtainable from the *twisted torus* $TT(2a+d, a)$ by trading a maximum of $2a$ edges for as many new edges, where $d = -1, +1$. To that extent, $TT(2a + d, a)$ may be viewed as a pivot from which to obtain $\mathcal{C}_{(2a+d)a}(1, (2a + d)k - 1)$ for all admissible k.

It turns out that each graph in each family is *tight-optimal* [10], hence its average distance is the least among all circulants of the same order and size.

1.1. Definitions and preliminaries

When we speak of a graph, we mean a finite, simple, undirected and connected graph, and write "G is isomorphic to H" as $G \cong H$. Let $\mathrm{dia}(G)$ represent the diameter of G [7].

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Table 1
Some characteristics of $\mathcal{C}_{(2a+d)a}(1, (2a+d)k-1),\ d = -1, 0, +1$.

	Bipartite?	Odd girth	Distance-wise vertex distribution
$\mathcal{C}_{(2a-1)a}\,(1, (2a-1)k-1)$	No	$2a-1$	$1+\underbrace{4i}_{1\le i\le a-1}+(a-1)$
$\mathcal{C}_{2a^2}\,(1, 2ak-1)$	Yes	–	$1+\underbrace{4i}_{1\le i\le a-1}+(2a-1)$
$\mathcal{C}_{(2a+1)a}\,(1, (2a+1)k-1)$	No	$2a+1$	$1+\underbrace{4i}_{1\le i\le a-1}+(3a-1)$

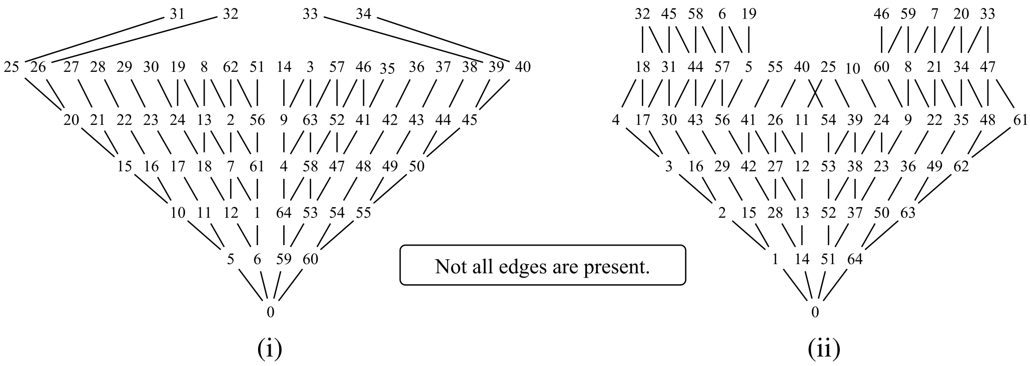


Fig. 1. Level diagrams of (i) $\mathcal{C}_{65}(5, 6)$ and (ii) $\mathcal{C}_{65}(1, 14)$.

Say that a vertex v is at *level i* relative to a fixed vertex u if the (shortest) *distance* between u and v is equal to i. Vertices at a distance of $\mathrm{dia}(G)$ from u are called *diametrical* relative to u. A *level diagram* of G relative to u consists of a layout of the graph in which vertices at a distance of i from u appear on a line at "height" i above u, for $0 \le i \le \mathrm{dia}(G)$. If G is known to be vertex transitive (a property held by a circulant), then the form of its level diagram is independent of the choice of the source vertex.

Let P_m denote the path having the vertex set $\{0, \ldots, m-1\},\ m \ge 2$, and let C_n denote the cycle having the vertex set $\{0, \ldots, n-1\},\ n \ge 3$. In each case, the adjacencies $\{i, i+1\}$ exist in a natural way. The *Cartesian product* $G\square H$ of graphs $G = (U, D)$ and $H = (W, F)$ is defined as follows: $V(G\square H) = U \times W$, and $E(G\square H) = \{\{(a, x), (b, y)\} \mid \{a, b\} \in D$ and $x = y$, or $\{x, y\} \in F$ and $a = b\}$ [7]. Whereas $P_m\square P_n$ is known as the $m \times n$ grid, $C_m\square C_n$ is known as the $m \times n$ torus.

A circulant in the present study connotes a four-regular circulant. To that end, let n, r, s be positive integers, where $n \ge 6$, and $1 \le r < s < n/2$. Then the circulant $\mathcal{C}_n(r, s)$ consists of the vertex set $\{0, \ldots, n-1\}$ and the edge set $\{\{i, i \pm r\},\ \{i, i \pm s\} \mid 0 \le i \le n-1\}$, where $i \pm r$ and $i \pm s$ are each taken modulo n. The parameters r and s are called the *step sizes*. If one of the step sizes is fixed at one, then the circulant is also known as a *chordal ring* or a *double-loop network*.

Proposition 1.1 ([4]). *The diameter of a four-regular circulant on n vertices is greater than or equal to* $\left\lceil \frac{1}{2}\left(-1+\sqrt{2n-1}\right)\right\rceil$. ■

A circulant, say G, is said to be *optimal* if its diameter meets the lower bound from Proposition 1.1 [2]. Meanwhile G may contain a maximum of $4i$ vertices at the ith level relative to a fixed vertex, $1 \le i \le \mathrm{dia}(G)$ [4], and if that bound is reached at each level from 1 to $\mathrm{dia}(G) - 1$, then G is said to be *tight-optimal* [10]. A tight-optimal circulant is necessarily optimal.

The graphs $\mathcal{C}_{65}(5, 6)$ and $\mathcal{C}_{65}(1, 14)$ appear in Fig. 1 to illustrate the foregoing. Whereas the two are optimal and of the same order/size, the former is tight-optimal, while the latter is not.

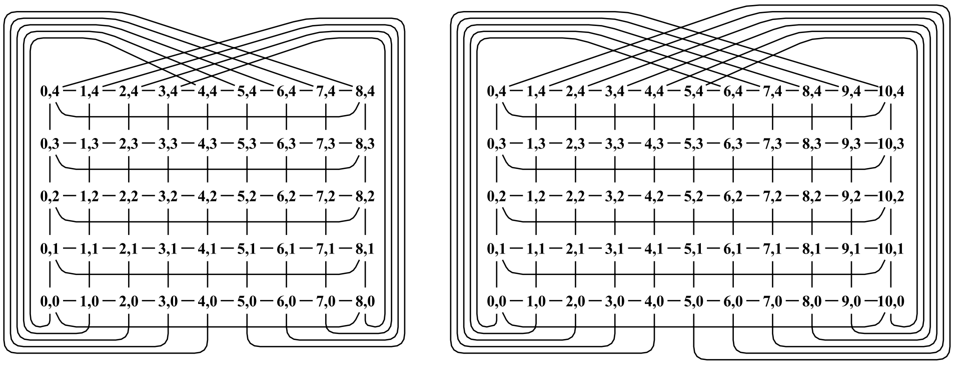
1.2. Twisted torus

For $a \ge 5$ and d in $\{-1, 0, 1\}$, the $(2a+d) \times a$ *twisted torus*, denoted by $TT(2a+d, a)$, is a variant of the torus $C_{2a+d}\square C_a$. Its vertex set is given by $\{(i, j) \mid 0 \le i \le 2a+d-1$ and $0 \le j \le a-1\}$, while its edge set consists of:

- $\{(i, j), (i+1, j)\} \mid 0 \le i \le 2a+d-2$ and $0 \le j \le a-1$, called the "horizontal" edges
- $\{(i, j), (i, j+1)\} \mid 0 \le i \le 2a+d-1$ and $0 \le j \le a-2$, called the "vertical" edges
- $\{(0, j), (2a+d-1, j)\} \mid 0 \le j \le a-1$, called the "wrap-around" edges, and
- $\{(i, 0), (i+a+d, a-1)\} \mid 0 \le i \le 2a+d-1$, called the "twisted" edges.

The arithmetic is modulo $2a+d$ in the first co-ordinate, and modulo a in the second. The following result is from B. Alspach [personal communication].

Lemma 1.2. *$TT(2a+d, a)$ is a Cayley graph on an abelian group.*



(i) (ii)

Fig. 2. (i) $TT(9,5)$ and (ii) $TT(11,5)$.

Proof. Consider the following mappings from the vertex set of this graph to itself:

- $\alpha(i,j) = (i+1, j)$, and
- $\beta(i,j) = \begin{cases}(i, j+1), & 0 \le j \le a-2\\ (i+a, 0), & j = a-1.\end{cases}$

It is easy to see that α and β are each a well-defined automorphism, and that they generate a vertex-transitive group. Also, they commute under composition, so they generate an abelian transitive group. Accordingly, $TT(2a+d, a)$ is a Cayley graph on the group generated by α and β. ■

It follows that $TT(2a+d, a)$ is vertex transitive. Meanwhile $TT(2a, a)$ is known as the *rectangular twisted torus* [5]. In the rest of the paper, we deal mainly with $TT(2a-1, a)$ and $TT(2a+1, a)$. See Fig. 2 for $TT(9,5)$ and $TT(11,5)$, where vertices (i,j) appear as i,j.

What follows: Section 2 presents the structures of the twisted tori, thus laying the foundation for their transformations into the respective circulants. The transformation from $TT(2a-1, a)$ into $\mathcal{C}_{(2a-1)a}(1, (2a-1)k-1)$ itself appears next. Section 4 is analogous to Section 3 in respect of $\mathcal{C}_{(2a+1)a}(1, (2a+1)k-1)$. Section 5 shows that the circulants admit a perfect dominating set under certain conditions, and they are largely nonisomorphic to those earlier devised by Beivide et al. [1]. Finally, Section 6 presents certain concluding remarks.

2. Structures of the twisted tori

2.1. $TT(2a-1, a)$

Let V_j denote the set of vertices at a distance of j from $(0,0)$ in $TT(2a-1, a)$, where $j \ge 0$. We build the sets V_j. To that end, let $A_0 = \{(0,0)\}$, $B_0 = C_0 = \emptyset$, and for $j \ge 1$, let A_j consist of vertices at a distance of j from $(0,0)$ relative to the grid $P_{2a-1}\Box P_a$; let B_j consist of vertices at a distance of $j-1$ from $(2a-2, 0)$ relative to the same grid; and let C_j consist of vertices at a distance of $j-1$ from $(a-1, a-1)$ relative to the same grid. Accordingly,

$$A_j = \{(i, j-i) : 0 \le i \le j\} \quad 1 \le j \le a-1$$

$$B_j = \begin{cases}\{(2a-1-j+i, i) : 0 \le i \le j-1\} & 1 \le j \le a-1\\ \{(j+i-1, i) : 1 \le i \le a-1\} & j = a\end{cases}$$

$$C_j = L_j \cup R_j,\ 1 \le j \le a-1,$$

where

$$L_j = \begin{cases}\emptyset & j = 1\\ \{(a-j+i, a-1-i) : 0 \le i \le j-2\} & 2 \le j \le a-1\end{cases}$$

$$R_j = \{(a-1+i, a-j+i) : 0 \le i \le j-1\} \quad 1 \le j \le a-1.$$

For $0 \le j \le a-1$, A_j, B_j and C_j are mutually disjoint, and $V_j = A_j \cup B_j \cup C_j$. For $1 \le j \le a-1$, $|A_j| = j+1$, $|B_j| = j$ and $|C_j| = 2j-1$, so $|V_j| = 4j$. Next, $V_a = B_a$. (Note that $|V_a| = a-1$.) Accordingly, the vertex distribution of $TT(2a-1, a)$ is given by $1 + \underbrace{4i}_{1\le i\le a-1} + (a-1)$.

It follows that the diameter of $TT(2a-1, a)$ is equal to a, and its average distance is equal to $\frac{1}{(2a-1)a}\Big(\big(\sum_{i=1}^{a-1} 4i^2\big) + (a-1)a\Big) \simeq \frac{4a-1}{6}$.

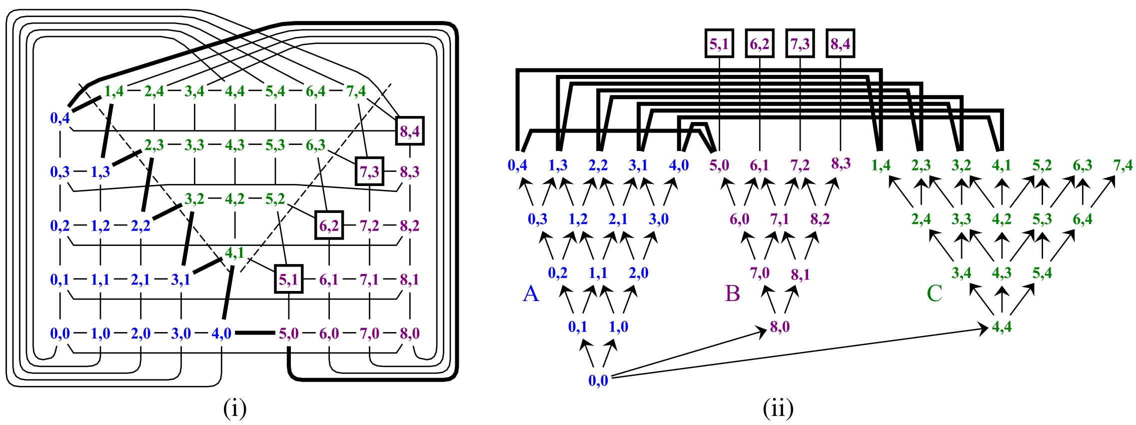


Fig. 3. $TT(9, 5)$ and its level diagram.

Fig. 3(i)/(ii) depicts the foregoing construction in respect of $TT(9, 5)$. Vertices diametrical with respect to (0, 0) appear within "rectangles". (The "dashed" lines are meant to delineate various vertex segments.) Not all edges are present in Fig. 3(ii). (The arrows highlight the progress of the shortest paths.) Interestingly, there exist a total of $2a$ edges that run between certain vertices at the $(a-1)$th level, and they form the following cycle:

$$(0, a-1) - (1, a-1) - (1, a-2) - (2, a-2) - \cdots - (a-1, 1) - (a-1, 0) - (a, 0) - (0, a-1).$$

The "dark" lines in Fig. 3 highlight these edges in respect of $TT(9, 5)$. Vertices at each of the remaining levels being mutually nonadjacent, the odd girth of $TT(2a-1, a)$ is equal to $2a-1$.

2.2. $TT(2a + 1, a)$

The discussion here is similar to that in Section 2.1, so the details will be trimmed. For $j \geq 0$, let W_j denote the set of vertices at a distance of j from (0, 0), and for $1 \leq j \leq a-1$, let

$$A_j = \begin{cases} \{(i, j-i) : 0 \leq i \leq j\} & 0 \leq j \leq a-1 \\ \{(i, j-i) : 1 \leq i \leq j\} & j = a \end{cases}$$

$$B_j = \begin{cases} \{(2a+1-j+i, i) : 0 \leq i \leq j-1\} & 1 \leq j \leq a-1 \\ \emptyset & j = a \end{cases}$$

$$C_j = L_j \cup R_j,\ 1 \leq j \leq a,$$

where

$$L_j = \begin{cases} \emptyset & j = 1 \\ \{(a+2-j+i, a-1-i) : 0 \leq i \leq j-2\} & 2 \leq j \leq a \end{cases}$$

$$R_j = \{(a+1+i, a-j+i) : 0 \leq i \leq j-1\} \quad 1 \leq j \leq a.$$

Check to see that $W_0 = \{(0, 0)\}$, and $W_j = A_j \cup B_j \cup C_j,\ 1 \leq j \leq a$, where $|W_j| = 4j,\ 1 \leq j \leq a-1$, and $|W_a| = 3a-1$. Accordingly, the vertex distribution of $TT(2a+1, a)$ is given by $1 + \underbrace{4i}_{1 \leq i \leq a-1} + (3a-1)$. It follows that the diameter of $TT(2a+1, a)$ is equal to a, and the average distance is equal to $\frac{1}{(2a+1)a}\left(\left(\sum_{i=1}^{a-1} 4i^2\right) + (3a-1)a\right) \simeq \frac{4a+1}{6}$.

Fig. 4 depicts the foregoing construction in respect of $TT(11, 5)$. Note that there exist a total of $2a$ edges that run between certain vertices at the ath level, and they form the following cycle:

$$(1, a-1) - (2, a-1) - (2, a-2) - (3, a-2) - \cdots - (a, 1) - (a, 0) - (a+1, 0) - (1, a-1).$$

The "dark" lines in Fig. 4(i)/(ii) highlight the foregoing edges in respect of $TT(11, 5)$. Vertices at each of the remaining levels being mutually nonadjacent, the odd girth of $TT(2a+1, a)$ is equal to $2a+1$.

3. From $TT(2a-1, a)$ to $\mathcal{C}_{(2a-1)a}(1, (2a-1)k-1)$

Table 2 presents a set of frequently used terms in this section. Note that $s < \frac{1}{2}n < n-s < n$, and that s is coprime with each of a, k and n. The method of attack is as follows: (i) set up a bijection from the vertex set of $TT(2a-1, a)$ to that of the circulant, and (ii) isolate those edges that come in the way of the graph being isomorphic to the circulant, and replace

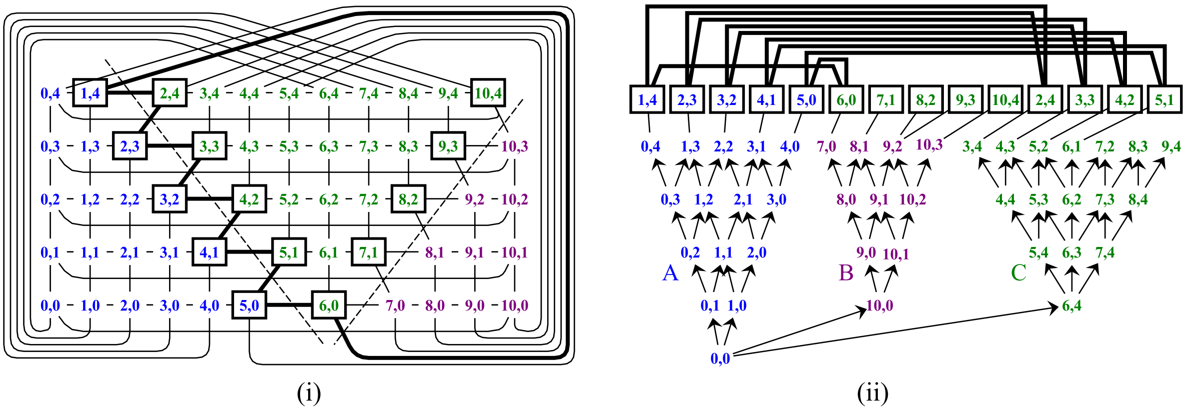
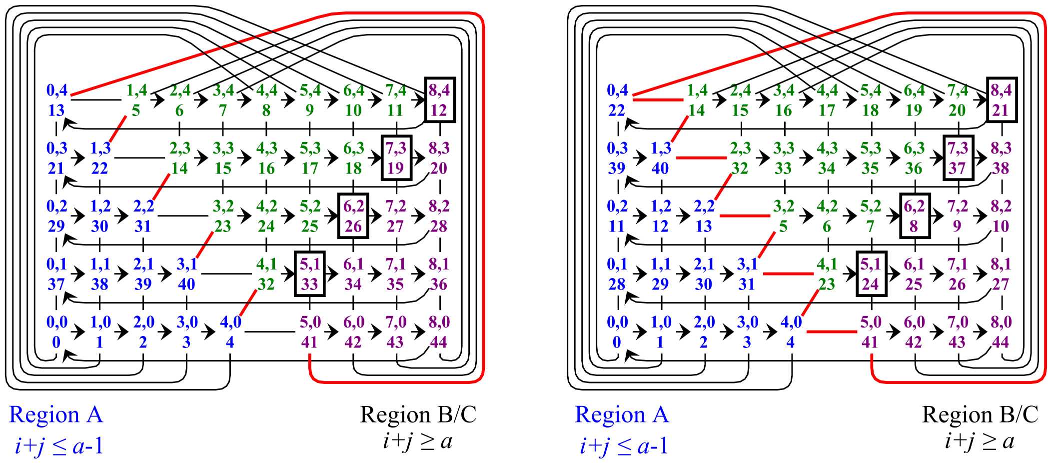


Fig. 4. $TT(11, 5)$ and its level diagram.

Table 2
Nomenclature in respect of Section 3.

a:	Parameter of $TT(2a-1, a)$ and $\mathcal{C}_{(2a-1)a}(1, (2a-1)k-1)$
n:	Number of vertices in $TT(2a-1, a)$/circulant, $n = (2a-1)a$
k:	Parameter of the circulant, $1 \le k \le \lfloor \frac{1}{2}(a-1) \rfloor$ with $\gcd(a, k) = 1$
k^{-1}:	Multiplicative inverse of k relative to a
s:	Non-unit step size of the circulant, $s = (2a-1)k-1$
f:	Bijection from $\{0, \ldots, 2a-2\} \times \{0, \ldots, a-1\}$ to $\{0, \ldots, (2a-1)a-1\}$
S_i:	Sets in the partition of $\{0, \ldots, (2a-1)a-1\}$, $\lvert S_i \rvert = 2a-1$, $0 \le i \le a-1$



(i) $k = 1, s = 8, n - s = 37$. (ii) $k = 2, s = 17, n - s = 28$.

Fig. 5. $TT(9, 5)$ in the light of Eq. (1).

them by as many new edges so that the resulting graph is the circulant. To that end, consider the following mapping from $V(TT(2a-1, a))$ to $\{0, \ldots, n-1\}$:

$$f(u, v) = \begin{cases} \big(u + (n-s)v\big) \mod n, & u + v \le a - 1 \\ \big(u + (n-s)v - (2a-1)\big) \mod n, & u + v \ge a \end{cases} \tag{1}$$

where $0 \le u \le 2a-2$ and $0 \le v \le a-1$. See Fig. 5 for its working on $TT(9, 5)$, where (u, v) and $f(u, v)$ coexist at each node. The "arrows" highlight the run of integer labels on each row, cf. Theorem 3.5. The relevance of the "dark red" lines will be clear shortly.

3.1. *f is a bijection*

Lemma 3.1. *$f(i, a-i) = ((2a-1)(ki-1) + a) \mod n,\ 1 \le i \le a$.*

Proof. By Eq. (1),

$$f(i, a-i) = (i + (n-s)(a-i) - (2a-1)) \mod n$$

$$\begin{aligned}
&= (i + n(a-i) - s(a-i) - (2a-1)) \mod n \\
&= (i - sa + si - (2a-1)) \mod n \\
&= (-sa + (s+1)i - (2a-1)) \mod n \\
&= (-((2a-1)k-1)a + (2a-1)ki - (2a-1)) \mod n \\
&= (-nk + a + (2a-1)(ki-1)) \mod n \\
&= ((2a-1)(ki-1) + a) \mod n. \quad \blacksquare
\end{aligned}$$

Corollary 3.2. (1) $f(a, 0) = n - a + 1 = (2a-1)(a-1) + a$.
(2) $f(i, a-i) = (2a-1)((ki) \mod a - 1) + a, \; 1 \le i \le a-1$.

Proof. (1) follows from Lemma 3.1 by a simple substitution and simplification. For (2), note that ki is not a multiple of a, since $\gcd(a, k) = 1$ and $0 < i < a$. In that light, let $ki = qa + r$ for some q and r, where $1 \le r \le a - 1$. By Lemma 3.1,

$$\begin{aligned}
f(i, a-i) &= ((2a-1)(ki-1) + a) \mod n \\
&= ((2a-1)(qa + r - 1) + a) \mod n \\
&= ((2a-1)qa + (2a-1)(r-1) + a) \mod n \\
&= (2a-1)(r-1) + a, \quad \text{since } n = (2a-1)a \text{ and } (2a-1)(r-1) + a < n \\
&= (2a-1)((ki) \mod a - 1) + a. \quad \blacksquare
\end{aligned}$$

Corollary 3.3. *If $1 \le i, j \le a$ and $i \ne j$, then $f(i, a-i)$ and $f(j, a-j)$ differ by a nonzero multiple of $2a - 1$.*

Proof. By Corollary 3.2, $f(i, a-i)$ is of the form $(2a-1)m + a$ for each i, so it suffices to show that $f(i, a-i) \ne f(j, a-j)$ if $i \ne j$.

Let $1 \le i, j \le a - 1$. By Corollary 3.2(2) $f(i, a-i)$ and $f(j, a-j)$ must be different, since $(ki) \mod a$ and $(kj) \mod a$ are different. By the same result, each of $f(i, a-i)$ and $f(j, a-j)$ is less than or equal to $(2a-1)(a-2) + a$, which itself is smaller than the remaining $f(a, 0) = (2a-1)(a-1) + a$. $\blacksquare$

Lemma 3.4. $f(0, j) = 1 + f(2a-2, j)$, $1 \le j \le a - 1$.

Proof. Note that

$$\begin{aligned}
&f(0, j) = ((n-s)j) \mod n, \quad \text{and} \\
&f(2a-2, j) = (2a - 2 + (n-s)j - (2a-1)) \mod n \\
&\qquad\qquad\;\; = ((n-s)j - 1) \mod n.
\end{aligned}$$

To complete the proof, it suffices to show that $f(0, j) > 0$, i.e., $(n-s)j$ is not a multiple of n. To that end, note that s and n are coprime and $s < n$, so $n - s$ and n too are coprime. Further, $0 < j < a$. The claim follows. $\blacksquare$

Theorem 3.5. *The mapping f in Eq. (1) is a bijection.*

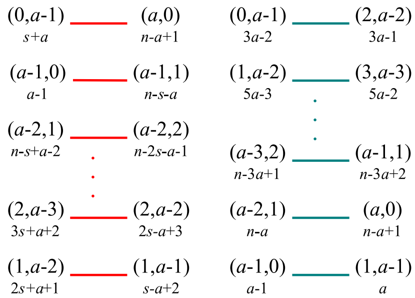
Proof. First note that $f(i, 0) = i, \; 0 \le i \le a-1$, and $f(a-1+i, 0) = n + i - a$, where $1 \le i \le a - 1$. For $j \ge 1$, observe that

- $f(i+1, j) = 1 + f(i, j), \; 0 \le i \le a - j - 2$
- $f(a-j, j) = f(a-j-1, j) - (2a-2)$
- $f(i+1, j) = 1 + f(i, j), \; a - j \le i \le 2a - 3$, and
- $f(0, j) = 1 + f(2a-2, j)$.

A simple reorganization shows that the following sequence

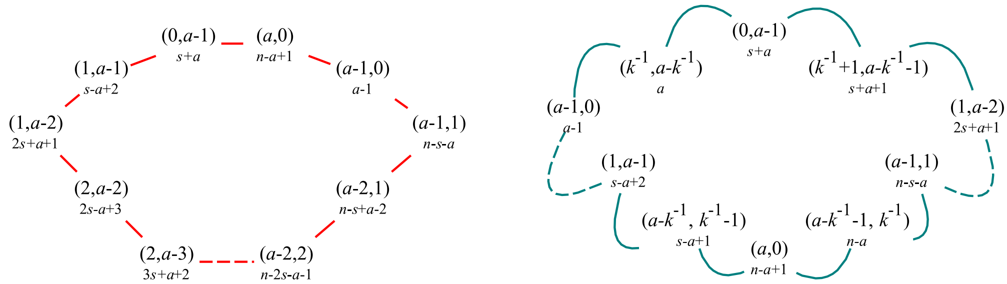
$$\langle f(a-j, j), f(a-j+1, j), \ldots, f(2a-2, j), f(0, j), f(1, j), \ldots, f(a-1-j, j) \rangle$$

constitutes a run of $2a - 1$ consecutive integers, i.e., labels on each row constitute a block of $2a - 1$ consecutive integers. (For $j = 0$, the integers are consecutive modulo n.) Corollary 3.3 ensures that the blocks from different rows are mutually exclusive. Further, the mapping itself is surjective. $\blacksquare$



(i) Incompatible edges. (ii) New edges.

Fig. 6. Incompatible edges vs. new edges: $k = 1$.



(i) Incompatible edges. (ii) New edges.

Fig. 7. Incompatible edges vs. new edges: $k \geq 2$.

3.2. Transformation continued

Call an edge $\{(i, j), (p, q)\}$ of $TT(2a-1, a)$ *compatible* if $|f(p, q)-f(i, j)|$ is equal to 1, s, $n-s$ or $n-1$, and call it *incompatible* otherwise.

Lemma 3.6. (1) *Let* $k = 1$. *Except for the* a *edges in the matching in* Fig. 6(i), *each edge in* $TT(2a-1, a)$ *is compatible.*
(2) *Let* $k \geq 2$. *Except for the* $2a$ *edges in the cycle in* Fig. 7(i), *each edge in* $TT(2a-1, a)$ *is compatible.*

Proof. We first attack (2).

A horizontal edge or a vertical edge running across Region A and Region B/C is excluded, and so is the twisted edge $\{(a, 0), (0, a-1)\}$. That the wrap-around edges and the remaining horizontal edges are compatible follows from the proof of Theorem 3.5. We argue (i) the remaining cases of vertical edges within each of the two regions, and (ii) the twisted edges.

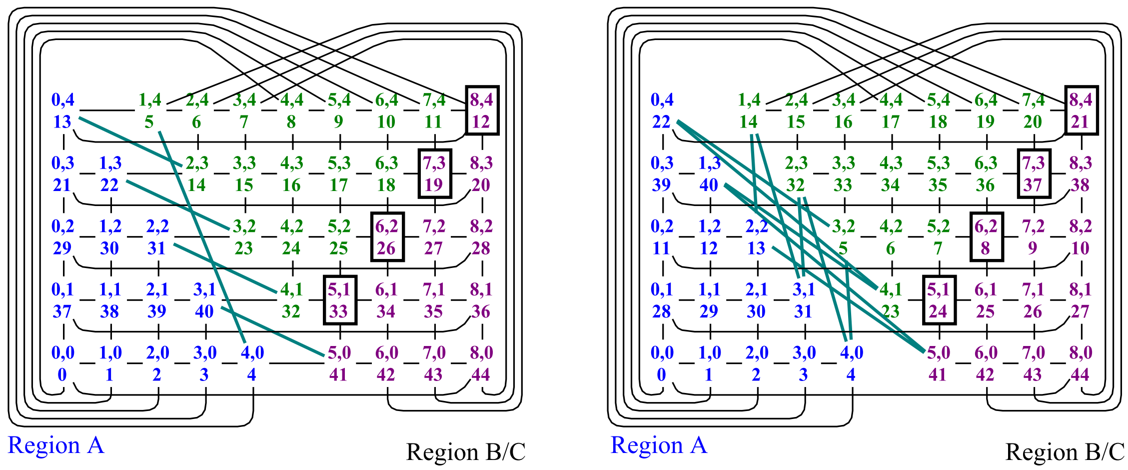
Let $\{(i, j), (i, j+1)\}$ be a vertical edge in Region A, and note that $f(i, j) = i+(n-s)j \bmod n$ and $f(i, j+1) = i+(n-s)(j+1) \bmod n$. Since $|\big(i+(n-s)j\big) - \big(i+(n-s)(j+1)\big)| = n-s$ that is positive and less than n, it follows that $|f(i, j)-f(i, j+1)|$ is equal to $n-s$ or s. The argument is similar if $\{(i, j), (i, j+1)\}$ is a vertical edge in Region B/C. We examine the remaining $2a-1$ (twisted) edges. (See Fig. 5(ii).)

- $\{(i, 0), (i+a-1, a-1)\}$, $0 \leq i \leq a-1$ (i.e., twisted edges across the two regions): check to see that $f(a-1, a-1) = s$, and $f(i+a-1, a-1) - f(i, 0) = f(a-1, a-1) - f(0, 0) = s$.
- $\{(i, 0), (i-a, a-1)\}$, $a+1 \leq i \leq 2a-2$ (i.e., twisted edges in Region B/C): note that $f(2a-2, 0) = n-1$ and $f(a-2, a-1) = s-1$, so $f(2a-2, 0)-f(a-2, a-1) = n-s$. Further, $f(i, 0)-f(i-a, a-1) = f(2a-2, 0)-f(a-2, a-1)$.

For (1), we need additionally show that the horizontal edges running across Region A and Region B/C conform to the stated condition. Such edges are: $\{(a-1-i, i), (a-i, i)\}$, $0 \leq i \leq a-1$. (Note that $s = 2a-2$ in this case.) By the proof of Theorem 3.5, $f(a-1-i, i) - f(a-i, i) = 2a-2$ if $1 \leq i \leq a-1$. Further, $|f(a-1, 0) - f(a, 0)| = n-2a+2$. (See Fig. 5(i).) ∎

Observe that the incompatible edges exist between vertices at the $(a-1)$th level relative to $(0, 0)$. They were distinguished in Section 2.1. The "red" lines in Fig. 5(i)/(ii) depict them in respect of $TT(9, 5)$. The next step is to trade them for as many new edges.

Lemma 3.7. 1. *If* $k = 1$, *then the graph obtainable from* $TT(2a-1, a)$ *by dropping the edges from* Fig. 6(i) *and adding the edges from* Fig. 6(ii) *is isomorphic to* $\mathcal{C}_{(2a-1)a}(1, 2a-2)$.
2. *If* $k \geq 2$, *then the graph obtainable from* $TT(2a-1, a)$ *by dropping the edges from* Fig. 7(i) *and adding the edges from* Fig. 7(ii) *is isomorphic to* $\mathcal{C}_{(2a-1)a}(1, (2a-1)k-1)$.



(i) $k = 1, s = 8, n - s = 37$. (ii) $k = 2, s = 17, n - s = 28$.

Fig. 8. $TT(9, 5)$ transformed into (i) $\mathcal{C}_{45}(1, 8)$ and (ii) $\mathcal{C}_{45}(1, 17)$.

Table 3
Nomenclature in respect of Section 4.

a:	Parameter of $TT(2a+1, a)$ and $\mathcal{C}_{(2a+1)a}(1, (2a+1)k-1)$
n:	Number of vertices in $TT(2a+1, a)$/circulant, $n = (2a+1)a$
k:	Parameter of the circulant, $1 \le k \le \lfloor \frac{1}{2}(a-1) \rfloor$ with $\gcd(a, k) = 1$
k^{-1}:	Multiplicative inverse of k relative to a
s:	Non-unit step size of the circulant, $s = (2a+1)k - 1$
g:	Bijection from $\{0, \ldots, 2a\} \times \{0, \ldots, a-1\}$ to $\{0, \ldots, (2a+1)a-1\}$
T_i:	Sets in the partition of $\{0, \ldots, (2a+1)a-1\}$, $\lvert T_i \rvert = 2a+1$ $0 \le i \le a-1$

Proof. The edges in Figs. 6(ii)/ 7(ii) are new in respect of $TT(2a-1, a)$, and the set of underlying vertices coincides with that in Figs. 6(i)/ 7(i). Further, check to see that each new edge is compatible. ■

Fig. 8 depicts the working of Lemma 3.7. The new edges appear as "dark green" lines. The edge exchanges themselves are such that the level diagram of $TT(2a-1, a)$ (cf. Section 2.1) holds true in respect of $\mathcal{C}_{(2a-1)a}(1, (2a-1)k-1)$ as well. Accordingly, the distance-wise level diagram, and hence the diameter, the average distance and the odd girth of the circulant coincide with those of $TT(2a-1, a)$ that appear in Section 2.1.

Theorem 3.8. *$\mathcal{C}_{(2a-1)a}(1, (2a-1)k-1)$ is a tight-optimal circulant.* ■

4. From $TT(2a+1, a)$ to $\mathcal{C}_{(2a+1)a}(1, (2a+1)k-1)$

The discussion here is similar to that in Section 3, so the details will be trimmed. See Table 3 for a set of frequently used terms in this section, and consider the following mapping from $V(TT(2a+1, a))$ to $\{0, \ldots, n-1\}$:

$$g(u, v) = \begin{cases} \big(u + (n-s)v\big) \mod n, & u + v \le a \\ \big(u + (n-s)v - (2a+1)\big) \mod n, & u + v \ge a+1 \end{cases} \tag{2}$$

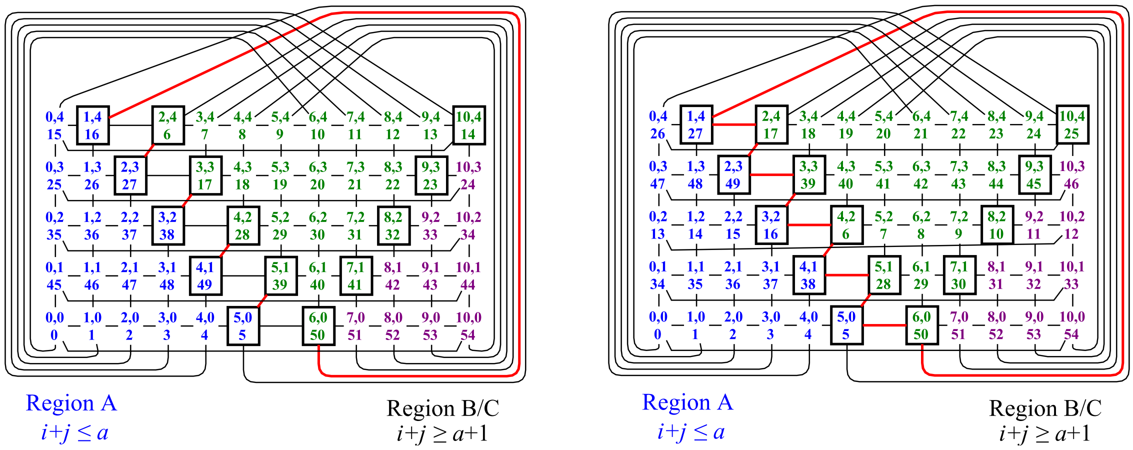
where $0 \le u \le 2a$ and $0 \le v \le a-1$. Fig. 9 illustrates its working on $TT(11, 5)$.

4.1. g is a bijection

Lemma 4.1. *$g(i+1, a-i) = ((2a+1)(ki-1) + (a+1)) \mod n$, $1 \le i \le a$.*

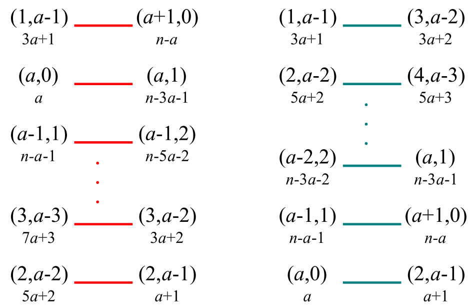
Proof. By Eq. (2),

$$\begin{aligned} g(i+1, a-i) &= (i + 1 + (n-s)(a-i) - (2a+1)) \mod n \\ &= (i + 1 + n(a-i) - s(a-i) - (2a+1)) \mod n \\ &= (i + 1 - sa + si - (2a+1)) \mod n \\ &= (1 - sa + (s+1)i - (2a+1)) \mod n \\ &= (1 - ((2a+1)k - 1)a + (2a+1)ki - (2a+1)) \mod n \\ &= (1 - nk + a + (2a+1)(ki-1)) \mod n \\ &= ((2a+1)(ki-1) + (a+1)) \mod n. \quad ■ \end{aligned}$$



(i) $k = 1, s = 10, n - s = 45$. (ii) $k = 2, s = 21, n - s = 34$.

Fig. 9. $TT(11, 5)$ in the light of Eq. (2).



(i) Incompatible edges. (ii) New edges.

Fig. 10. Incompatible edges vs. new edges: $k = 1$.

Corollary 4.2. (1) $g(i + 1, a - i) = (2a + 1)((ki) \mod a - 1) + (a + 1),\ 1 \le i \le a - 1$.
(2) $g(a + 1, 0) = 2a^2 = (2a + 1)(a - 1) + (a + 1)$.

Proof. (1) follows by an argument as in the proof of Corollary 3.2(2), while (2) follows from Lemma 4.1 by a simple substitution and simplification. ∎

The claims 4.3, 4.4 and 4.5 below are analogous to the claims 3.3, 3.4 and 3.5, respectively. Their proofs are omitted.

Corollary 4.3. *If* $1 \le i, j \le a$ *and* $i \ne j$*, then* $g(i + 1, a - i)$ *and* $g(j + 1, a - j)$ *differ by a nonzero multiple of* $2a + 1$. ∎

Lemma 4.4. $g(0, j) = 1 + g(2a, j),\ 1 \le j \le a - 1$. ∎

Theorem 4.5. *The mapping g in Eq. (2) is a bijection.* ∎

4.2. Transformation continued

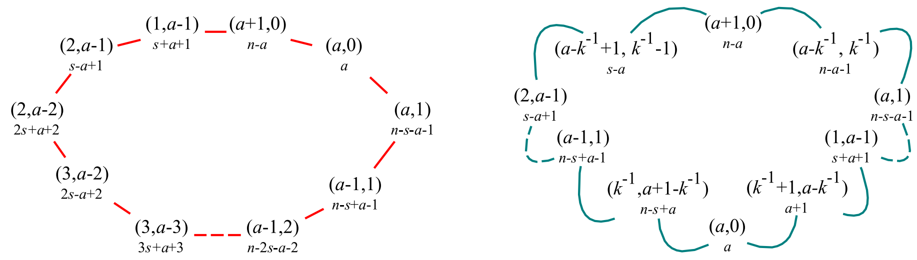
Call an edge (in)compatible as in Section 3.

Lemma 4.6. 1. *Let* $k = 1$*. Except for the edges in the matching in Fig. 10(i), each edge in* $TT(2a + 1, a)$ *is compatible.*
2. *Let* $k \ge 2$*. Except for the edges in the cycle in Fig. 11(i), each edge in* $TT(2a + 1, a)$ *is compatible.*

Proof. Similar to the proof of Lemma 3.6. ∎

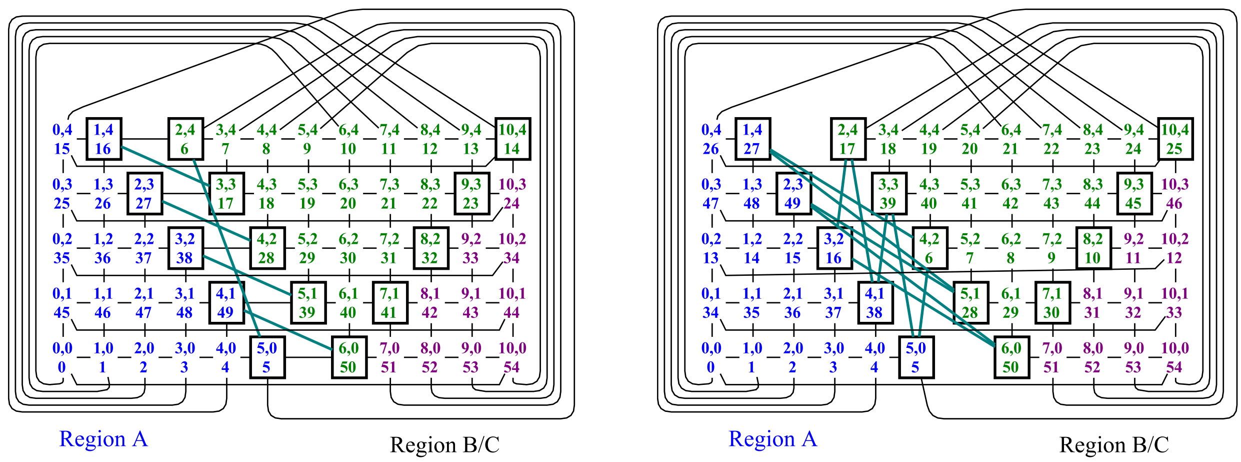
The incompatible edges exist between vertices at the *a*th level relative to (0, 0). They were distinguished in Section 2.2. (The "red" lines in Fig. 9(i)/(ii) depict them in respect of $TT(11, 5)$.) The next step is to trade them for as many new edges.

Lemma 4.7. 1. *If* $k = 1$*, then the graph obtainable from* $TT(2a + 1, a)$ *by dropping the edges from Fig. 10(i) and adding the edges from Fig. 10(ii) is isomorphic to* $\mathcal{C}_{(2a+1)a}(1, 2a)$.



(i) Incompatible edges. (ii) New edges.

Fig. 11. Incompatible edges vs. new edges: $k \geq 2$.



(i) $k = 1, s = 10, n - s = 45$. (ii) $k = 2, s = 21, n - s = 34$.

Fig. 12. $TT(11, 5)$ transformed into (i) $\mathcal{C}_{55}(1, 10)$ and (ii) $\mathcal{C}_{55}(1, 21)$.

2. *If* $k \geq 2$, *then the graph obtainable from* $TT(2a + 1, a)$ *by dropping the edges from* Fig. 11(i) *and adding the edges from* Fig. 11(ii) *is isomorphic to* $\mathcal{C}_{(2a+1)a}(1, (2a + 1)k - 1)$. ■

Fig. 12 depicts the working of Lemma 4.7. The new edges appear as "dark green" lines. The edge exchanges themselves are such that the level diagram of $TT(2a + 1, a)$ (cf. Section 2.2) holds true in respect of $\mathcal{C}_{(2a+1)a}(1, (2a + 1)k - 1)$ as well. Accordingly, the distance-wise level diagram, and hence the diameter, the average distance and the odd girth of the circulant coincide with those of $TT(2a + 1, a)$ that appear in Section 2.2.

Theorem 4.8. $\mathcal{C}_{(2a+1)a}(1, (2a + 1)k - 1)$ *is a tight-optimal circulant.* ■

5. Some related issues

The present section shows that the circulants admit a perfect dominating set under certain conditions on a and k, and that they are largely nonisomorphic to those previously devised by Beivide et al. [1]. As earlier, a and k are such that $a \geq 5$; $\gcd(a, k) = 1$; and $1 \leq k \leq \lfloor \frac{1}{2}(a - 1) \rfloor$.

5.1. Perfect dominating sets

A vertex subset S of a graph G is said to be a *perfect dominating set* (or an *efficient dominating set*) if the *balls* of radius one centered at various vertices in S constitute a partition of $V(G)$ [11,9]. The concept has applications in areas such as perfect codes and resource placement. Meanwhile the following result is relevant in the present study.

Theorem 5.1 ([11]). *A circulant* $\mathcal{C}_n(r, s)$ *admits a perfect dominating set if and only if* $n \equiv 0 \pmod 5$, *and* $r, s, |r \pm s| \not\equiv 0 \pmod 5$. ■

Corollary 5.2. (1) *If* $a \equiv 0 \pmod 5$, *and either* $k \equiv 1 \pmod 5$ *or* $k \equiv 2 \pmod 5$, *then* $\mathcal{C}_{(2a-1)a}(1, (2a - 1)k - 1)$ *admits a perfect dominating set.*

(2) *If* $a \equiv 0 \pmod 5$, *and either* $k \equiv 3 \pmod 5$ *or* $k \equiv 4 \pmod 5$, *then* $\mathcal{C}_{(2a+1)a}(1, (2a + 1)k - 1)$ *admits a perfect dominating set.*

Proof. First consider (1), and note that $(2a-1)k \not\equiv 0 \pmod 5$, since $a \equiv 0 \pmod 5$ and $\gcd(a, k) = 1$. Next, $(2a-1)k-1 = 2ak-(k+1)$ and $(2a-1)k-2 = 2ak-(k+2)$, none of which is a multiple of 5, since $k \equiv 1 \pmod 5$ or $k \equiv 2 \pmod 5$. The claim follows by an invocation of Theorem 5.1. The argument is similar for (2). ■

5.2. (Non)isomorphism vis-à-vis other circulants

Beivide et al. [1] presented a class of circulants, called *midimew networks*. Among them, the following are similar to the ones in the present paper: $\mathcal{C}_{(2a-1)a}(a-1, a)$ and $\mathcal{C}_{(2a+1)a}(a, a+1)$. The following questions arise:

- Is $\mathcal{C}_{(2a-1)a}(a-1, a) \cong \mathcal{C}_{(2a-1)a}(1, (2a-1)k-1)$ for some k?
- Is $\mathcal{C}_{(2a+1)a}(a, a+1) \cong \mathcal{C}_{(2a+1)a}(1, (2a+1)k-1)$ for some k?

It turns out that the answer to the first question is in the negative, and the answer to the second is in the affirmative for $k = 1$. To that end, the following result is useful.

Theorem 5.3 (*[6]*).
(1) *If* $\gcd(r, n) = 1$, *then* $\mathcal{C}_n(r, s) \cong \mathcal{C}_n(1, r^{-1}s \mod n)$.
(2) *If* $s_1 \neq s_2,\ s_1 \neq n - s_2$, *and* $s_1 s_2 \not\equiv \pm 1 \pmod n$, *then* $\mathcal{C}_n(1, s_1) \not\cong \mathcal{C}_n(1, s_2)$. ■

Lemma 5.4. $\mathcal{C}_{(2a-1)a}(a-1, a) \not\cong \mathcal{C}_{(2a-1)a}(1, (2a-1)k-1)$ *for any* k.

Proof. Let $n = (2a-1)a$, and note that $(a-1)^{-1} = 2a^2-3a-1$, and $\left((2a^2-3a-1)a \mod n\right) = n-2a$. By Theorem 5.3(1), $\mathcal{C}_{(2a-1)a}(a-1, a) \cong \mathcal{C}_{(2a-1)a}(1, n-2a)$, i.e., $\mathcal{C}_{(2a-1)a}(1, 2a)$. It is clear that $2a \neq (2a-1)k-1$ for any $a \geq 5$. Also, $n-2a = 2a^2-3a \neq (2a-1)k-1$ for $1 \leq k \leq \lfloor \frac{1}{2}(a-1) \rfloor$. Further, it is easy to check that $2a((2a-1)k-1) \not\equiv \pm 1 \pmod n$. The claim follows by Theorem 5.3(2). ■

Lemma 5.5. $\mathcal{C}_{(2a+1)a}(a, a+1) \cong \mathcal{C}_{(2a+1)a}(1, (2a+1)k-1)$ *if and only if* $k = 1$.

Proof. Let $n = (2a+1)a$, and note that $(a+1)^{-1} = 2a^2-a+1$, and $\left((2a^2-a+1)a \mod n\right) = 2a$. By Theorem 5.3(1), $\mathcal{C}_{(2a+1)a}(a, a+1) \cong \mathcal{C}_{(2a+1)a}(1, 2a)$ that coincides with $\mathcal{C}_{(2a+1)a}(1, (2a+1)k-1)$ if $k = 1$.

For the converse, first note that $2a \neq (2a+1)k-1$ for $k \geq 2$, and $n-2a = 2a^2-a \neq (2a+1)k-1$ for any k. Further, it is easy to see that $2a((2a+1)k-1) \not\equiv \pm 1 \pmod n$. The claim follows by Theorem 5.3(2). ■

6. Concluding remarks

This paper systematically accentuates and amplifies two major families of optimal circulants, viz., $\mathcal{C}_{(2a-1)a}(1, (2a-1)k-1)$ and $\mathcal{C}_{(2a+1)a}(1, (2a+1)k-1)$, devised by Tzvieli [13], where $a \geq 5$; $1 \leq k \leq \lfloor \frac{1}{2}(a-1) \rfloor$; and $\gcd(a, k) = 1$. A major finding is that each circulant in each family is tight-optimal, hence its average distance is the least among all circulants of the same order and size. The scheme for that purpose consists of a careful transformation of the twisted torus $TT(2a+d, a)$ into $\mathcal{C}_{(2a+d)a}(1, (2a+d)k-1)$, where $d = -1, +1$.

It is known that a four-regular circulant of diameter a may contain a maximum of $2a^2+2a+1$ vertices [3,4]. In that light, $\mathcal{C}_{(2a+1)a}(1, (2a+1)k-1)$ may be viewed as a circulant that is nearly dense.

It turns out that the odd girth of each circulant is approximately equal to $\sqrt{2n}$, where n denotes its order. (A large odd girth is a plus in a network.) Further, if $a \equiv 0 \pmod 5$ and certain conditions are imposed on k, then the graph admits a perfect dominating set.

Acknowledgment

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