PERGAMON

Applied Mathematics Letters 13 (2000) 51–55

Applied Mathematics Letters

www.elsevier.nl/locate/aml

Distance Regularity in Direct-Product Graphs

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(Received March 1999; accepted April 1999)

Abstract—The direct product (also called Kronecker product, tensor product, and cardinal product) $G \times H$ of distance-regular graphs is investigated. It is demonstrated that the product is distance-regular only when G and H are very restricted distance-regular graphs.

Keywords—Distance-regular graphs, Direct product.

1. INTRODUCTION

Every connected graph is representable by means of a *level diagram* (cf. [1]) as follows. Choose a vertex u, and let it be the sole resident of level zero. The vertices on level i are precisely those whose distance from u is i. Now add edges of the graph and note that the edges occur only between vertices of adjacent levels and among vertices of the same level. *Distance regularity* is definable in terms of a level diagram. Let d be the diameter of a given graph G, and let v be a vertex of G on level i. Further, let

$$a_i = \text{number of vertices on level } i \text{ adjacent to } v,\ i = 1, \dots, d,$$
$$b_i = \text{number of vertices on level } i+1 \text{ adjacent to } v,\ i = 0, \dots, d-1,$$
$$c_i = \text{number of vertices on level } i-1 \text{ adjacent to } v,\ i = 1, \dots, d.$$

We got to know about the unique characteristic of the (3, 12)-cage from C. Godsil, and received much-needed encouragement from P. Weichsel. We are also thankful to the referee, whose comments on the earlier draft led to an improvement in the paper.

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0893-9659/1999/$ - see front matter. Typeset by AMS-TEX
PII: S0893-9659(99)00144-5

G is said to be *distance-regular* if it is connected and the numbers a_i, b_i, and c_i depend only on i and not on the choice of the level diagram or on the choice of v. A distance-regular graph is necessarily regular. On the other hand, every regular graph need not be distance-regular. Cycles, complete graphs, and hypercubes are some of the familiar graphs easily seen to be distance-regular. Brouwer *et al.* [2] list several characteristics and applications of this class of graphs.

What distance-regular graphs survive stress of the direct product? It turns out that most such graphs are of low diameter. In particular, if G and H are distance-regular graphs with $d(G) > 2$ and $d(H) > 2$, then $G \times H$ is not distance-regular. (Here $d(G)$ denotes the diameter of G.) This is to be contrasted with Weichsel's result with respect to the Cartesian product [3].

Let G be a distance-regular graph, and let Δ be the degree of G. It is easy to see that $b_0 = \Delta$ and $c_1 = 1$. Further, if G is bipartite, then $a_i = 0$ for all i and $c_d = \Delta$. The pair of sequences $[(b_0, \dots, b_{d-1}); (c_1, \dots, c_d)]$ is called the *intersection array*. It contains all the essential information about the graph but falls short of uniquely determining the graph.

By a graph is meant a finite, simple, and undirected graph. Unless indicated otherwise, graphs are also connected and have at least two vertices. Let $G = (V, E)$ and $H = (W, F)$ be graphs. The *direct product* $G \times H$ of G and H is defined as follows: $V(G \times H) = V \times W$ and $E(G \times H) = \{\{(u, x), (v, y)\} : \{u, v\} \in E \text{ and } \{x, y\} \in F\}$. This product is variously known as Kronecker product, tensor product, and cardinal product. Certain relevant characteristics are as follows:

(i) $G \times H$ is bipartite iff G or H is bipartite;
(ii) $G \times H$ is connected iff G or H is nonbipartite;
(iii) if G and H are both bipartite, then $G \times H$ consists of two connected components; and
(iv) if $G = (V_0 \cup V_1, E)$ is a bipartite graph equipped with an automorphism that swaps the two colors, then for every bipartite graph H, the two components of $G \times H$ are isomorphic to each other [4].

Note that the $(3, 12)$-cage is bipartite and distance-regular, yet it does not admit of an automorphism swapping the two colors [2].

2. RESULTS

Our important result (that appears in Corollary 2.3 below) is that if $d(G) > 2$ and $d(H) > 2$, then $G \times H$ is not distance-regular. Theorem 2.4 is a characterization for distance-regularity of $G \times H$ where $d(G) = d(H) = 2$. Finally, we deal with the case when one or each of G and H is a complete graph and state certain results relating to $G \times K_2$.

THEOREM 2.1. *Let G and H be distance-regular graphs such that G or H is nonbipartite, and $d(G) \geq 2$, $d(H) \geq 2$. For $2 \leq k \leq \min\{d(G), d(H)\}$, if $c_k(G) \neq \Delta(G)$ or $c_k(H) \neq \Delta(H)$, then $G \times H$ is not distance-regular.*

PROOF. Let G, H, and k be as stated, and note that $G \times H$ is connected. For a vertex u of G, consider the level diagram of G with u at level zero. Similarly, for a vertex v of H, consider the level diagram of H with v at level zero. We construct the level diagram of $G \times H$ with vertex (u, v) at level zero.

We first claim that for each k, there exist vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) at level k of $G \times H$ such that

(a) $\text{dist}_G(u, x_1) = k = \text{dist}_H(v, y_1)$,
(b) $\text{dist}_G(u, x_2) = k > \text{dist}_H(v, y_2)$, and
(c) $\text{dist}_G(u, x_3) < k = \text{dist}_H(v, y_3)$, where $\text{dist}_G(u, x_i)$ and $\text{dist}_H(v, y_i)$ are of the same parity, $1 \leq i \leq 3$.

For (a), note that G is xance-regular, and $2 \leq k \leq \min\{d(G), d(H)\}$, hence there exist vertices x_1 and y_1 in G and H, respectively, such that $\text{dist}_G(u, x_1) = k = \text{dist}_H(v, y_1)$. Clearly, $\text{dist}_{G \times H}((u, v), (x_1, y_1)) = k$.

For (b), let $u - u_1 - u_2 - \cdots - u_k$ be a shortest path of length k in G, and let v_1 be a vertex adjacent to v in H. If k is even, then $(u,v) - (u_1,v_1) - (u_2,v) - \cdots - (u_{k-1},v_1) - (u_k,v)$ is a shortest path of length k in $G \times H$. On the other hand, if k is odd, then $(u,v) - (u_1,v_1) - (u_2,v) - \cdots - (u_{k-1},v) - (u_k,v_1)$ is a shortest path of length k in $G \times H$. Letting $x_2 = u_k$ and $y_2 = v$ if k is even, and letting $x_2 = u_k$ and $y_2 = v_1$ if k is odd, we observe:

(i) $\text{dist}_G(u,x_2) = k > \text{dist}_H(v,y_2)$, and
(ii) $\text{dist}_G(u,x_2)$ and $\text{dist}_H(v,y_2)$ are of the same parity.

Argument for (c) is analogous to the foregoing.

For each of the vertices of the type (x_1,y_1), (x_2,y_2), and (x_3,y_3) at level k of $G \times H$ mentioned above, we compute c_k.

- Vertex (x_1,y_1) is adjacent to a vertex (r_1,s_1) at level $k-1$ iff $\text{dist}_G(u,r_1) = k-1 = \text{dist}_H(v,s_1)$, and x_1,r_1 (respectively, y_1,s_1) are adjacent in G (respectively, H). The number of such vertices at level $k-1$ of $G \times H$ is exactly $c_k(G) \cdot c_k(H)$.
- Vertex (x_2,y_2) is adjacent to a vertex (r_2,s_2) at level $k-1$ iff $\text{dist}_G(u,r_2) = k-1$, and x_2,r_2 are adjacent in G. The number of such vertices at level $k-1$ of $G \times H$ is exactly $c_k(G) \cdot \Delta(H)$.
- Vertex (x_3,y_3) is adjacent to a vertex (r_3,s_3) at level $k-1$ iff $\text{dist}_H(v,s_3) = k-1$, and y_3,s_3 are adjacent in H. The number of such vertices at level $k-1$ of $G \times H$ is exactly $\Delta(G) \cdot c_k(H)$.

If the graph $G \times H$ is to be distance-regular, then $c_k(G) \cdot c_k(H) = c_k(G) \cdot \Delta(H) = \Delta(G) \cdot c_k(H)$. This implies that $c_k(G) = \Delta(G)$ and $c_k(H) = \Delta(H)$. ∎

THEOREM 2.2. *If G and H are bipartite distance-regular graphs with $d(G) \geq 2$ and $d(H) \geq 2$, then each component of $G \times H$ is distance-regular iff $c_k(G) = \Delta(G)$ and $c_k(H) = \Delta(H)$, where $2 \leq k \leq \min\{d(G), d(H)\}$.*

PROOF. Let G and H be as stated. $G \times H$ consists of two (bipartite) components, where vertices (u,v) and (x,y) belong to the same component iff $\text{dist}_G(u,x)$ and $\text{dist}_H(v,y)$ are of the same parity. Observe that $d(G \times H) = \max\{d(G), d(H)\}$, cf. [5].

We construct a level diagram of one component of $G \times H$ as in the proof of Theorem 2.1. With vertex (u,v) at level zero, we distinguish among three types of vertices, namely, (x_1,y_1), (x_2,y_2) and (x_3,y_3), at level k, where

(a) $\text{dist}_G(u,x_1) = k = \text{dist}_H(v,y_1)$,
(b) $\text{dist}_G(u,x_2) = k > \text{dist}_H(v,y_2)$, and
(c) $\text{dist}_G(u,x_3) < k = \text{dist}_H(v,y_3)$.

Note that $\text{dist}_G(u,x_i)$ and $\text{dist}_H(v,y_i)$ are necessarily of the same parity, $1 \leq i \leq 3$.

By an argument as in the proof of Theorem 2.1, if each component of $G \times H$ is distance-regular, then $c_k(G) = \Delta(G)$ and $c_k(H) = \Delta(H)$, where $2 \leq k \leq \min\{d(G), d(H)\}$.

For the converse, assume that $c_k(G) = \Delta(G)$ and $c_k(H) = \Delta(H)$ for $2 \leq k \leq \min\{d(G), d(H)\}$. This means that each of G and H is of the form $K_{n,n}$. In this case, each component of $G \times H$ is again such. ∎

COROLLARY 2.3. *If G and H are distance-regular graphs with $d(G) > 2$ and $d(H) > 2$, then $G \times H$ or a component of $G \times H$ is not distance-regular.*

PROOF. Let G and H be as stated. Since $b_2(G) > 0$ for any distance-regular graph with diameter at least three, it is clear that $c_2(G) < \Delta(G)$. By Theorems 2.1 and 2.2, $G \times H$ or a connected component of $G \times H$ is not distance-regular. ∎

THEOREM 2.4. *Let G and H be distance-regular graphs with $d(G) = d(H) = 2$. Each component of $G \times H$ is distance-regular iff both G and H are bipartite.*

PROOF. Let G and H be as stated. First, assume that G is nonbipartite, in which case either $a_1(G) > 0$ or $a_2(G) > 0$. If $a_2(G) > 0$, then $c_2(G) < \Delta(G)$, and by Theorem 2.1, $G \times H$ is not distance-regular. Suppose that $a_1(G) > 0$. Let $u, x \in V(G)$, $v, y \in V(H)$, where $\text{dist}_G(u,x) = 1$ and $\text{dist}_H(v,y) = 2$. Now consider the level diagram of $G \times H$ with (u,v) at level zero. It is easy to see that (u,y) is at level two. Further, since $a_1(G) > 0$, the vertex (x,y) is also at level two. However, the number of common neighbors of (u,v) and (u,y) is $\Delta(G) \cdot \Delta(H)$, while the number of common neighbors of (u,v) and (x,y) is $a_1(G) \cdot \Delta(H)$. Since $a_1(G) < \Delta(G)$, it follows that $G \times H$ is not distance-regular. The converse follows from Theorem 2.2. ∎

The reader may check to see that results 2.1, 2.2, 2.3, and 2.4 lead to the following. If G and H are distance-regular graphs of diameter at least two, then $G \times H$ or a component of $G \times H$ is distance-regular iff each of G and H is isomorphic to $K_{n,n}$ for some n.

By an argument as in the proof of Theorem 2.4, if G is a distance-regular graph with $d(G) \geq 2$ and $n \geq 3$, then $G \times K_n$ is not distance-regular. The following result takes care of $K_m \times K_n$.

THEOREM 2.5. *For $m, n \geq 3$, $K_m \times K_n$ is distance-regular iff $m = n$.*

PROOF. Let $m, n \geq 3$, and consider the graph $K_m \times K_n$ which is a nonbipartite, regular graph of diameter two and degree $(m-1) \cdot (n-1)$. Let us examine the level diagram of $K_m \times K_n$ with vertex $(0,0)$ at level zero.

Vertices at level one are of the form (i,j), where $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$. Thus, this level has a total of $(m-1) \cdot (n-1)$ vertices. The remaining $m+n-2$ vertices are at level two, and are of the form $(i,0)$ and $(0,j)$, where $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$.

A vertex (i,j) at level one is adjacent to a vertex (p,q) at the same level iff

(i) $1 \leq p \leq m-1$, $p \neq i$, and
(ii) $1 \leq q \leq n-1$, $q \neq j$.

It follows that $a_1 = (m-2) \cdot (n-2)$. Since $c_1 = 1$, we have $b_1 = m+n-4$. (Recall that $a_i + b_i + c_i = \Delta$.)

Next examine adjacency among elements at level two. If a vertex is of the form $(i,0)$ (respectively, $(0,j)$), then it has a total of $n-1$ (respectively, $m-1$) neighbors on that level. Based on this, we have a_2 and c_2 in Table 1.

Table 1.

	For a vertex of the form $(i,0)$, $1 \leq i \leq m-1$	For a vertex of the form $(0,j)$, $1 \leq j \leq n-1$
a_2	$n-1$	$m-1$
c_2	$(m-2) \cdot (n-1)$	$(m-1) \cdot (n-2)$

It follows that the numbers a_k, b_k, and c_k depend only on k and not on the choice of a vertex (i,j) iff $m = n$. Note also that the level diagram itself is independent of the choice of the vertex at level 0. ∎

Dealing with $G \times K_2$

In the rest of the paper, we present certain remarks with respect to $G \times K_2$. This graph (that is connected iff G is nonbipartite) has been called *bipartite double* of G by Brouwer *et al.* who present a characterization for its distance-regularity and other related results [2, pp. 24–26]. In particular, they prove the following.

(1) $G \times K_2$ is distance-regular of diameter $2d+1$ iff G is distance-regular with $a_i = 0$ $(i < d)$ and $a_d > 0$. In this case, $G \times K_2$ is an antipodal 2-cover of G.
(2) If G is distance-regular and $j = \min\{i \mid a_i \neq 0\} < d$, then $G \times K_2$ is distance-regular iff $d = 2j$, $a_j = c_{j+1}$, $b_{j-i} = c_{j+i+1} = c_{j+i} + a_{j+i}$ $(i = 1, \ldots, j-1)$; if this is the case, then $d(G \times K_2) = 2j + 1 = d + 1$.

Odd cycles, complete graphs (on at least three vertices), Petersen graph, and Hoffman-Singleton graph are certain examples that satisfy conditions in (1), while Shrikhande graph with intersection array $[(6,3);(1,2)]$ and Clebsch graph (that is isomorphic to the halved 5-cube) are examples that satisfy conditions in (2).

An Example

There exists a graph G such that G is not distance-regular, yet $G \times K_2$ is distance-regular. To see this, consider the graph that appears in Figure 1. The reader may check to see that

(i) G is not distance-regular, and
(ii) $G \times K_2$ is isomorphic to Q_4 (appearing in Figure 2) that is known to be distance-regular.

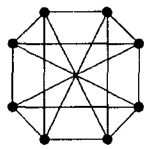

Figure 1. Graph G.

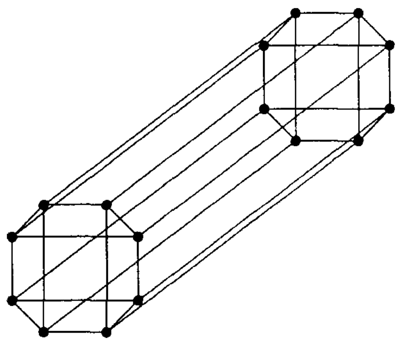

Figure 2. The graph Q_4.

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