The matrix S in Theorem 3 is obtained as

$$S = \begin{pmatrix} \frac{3}{4} & -1 \\ -1 & \frac{3}{4} \end{pmatrix}$$

Since the system equations discussed in [15] require $c_i = 1 (i = 1, 2)$, the condition given in [15] is not applied. In addition, we can also see that S is not a nonsingular M matrix, so the condition given in [15] does not also hold for the matrices given above.

III. Conclusions

A set of criteria have been derived ensuring the global asymptotic stability of delayed cellular neural networks (DCNNs) with more general output functions by introducing ingeniously infinitely real parameters, constructing suitable Lyapunov functionals and applying some analysis techniques, and these criteria are independent of delays and possess infinitely adjustable real parameters $w_i > 0$, α^*_{ij}, β^*_{ij}, η^*_{ij}, ζ^*_{ij}, α_{ij}, β_{ij}, η_{ij}, ζ_{ij} are any real numbers with $\alpha^*_{ij} + \beta^*_{ij} = 1$, $\alpha_{ij} + \beta_{ij} = 1$, $\eta^*_{ij} + \zeta^*_{ij} = 1$, $\eta_{ij} + \zeta_j = 1$ $(i, j = 1, 2, \ldots, n)$. These are of prime importance and great interest in many application fields and the design of networks. In addition, the methods of this paper may extended to discuss more complicated systems such as Hopfield neural networks(HNNs) and bi-directional associative memory (BAM) networks.

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Optimal $L(2, 1)$-Labeling of Strong Products of Cycles

Pranava K. Jha

***Abstract*—The $L(2, 1)$-labeling of a graph is an abstraction of assigning integer frequencies to radio transmitters such that i) transmitters that are one unit of distance apart receive frequencies that differ by at least two, and ii) transmitters that are two units of distance apart receive frequencies that differ by at least one. The least span of frequencies in such a labeling is referred to as the λ-number of the graph. It is shown that if $k \geq 1$ and $m_0, \ldots, m_{k-1}$ are each a multiple of $3^k + 2$, then $\lambda(C_{m_0} \boxtimes \ldots \boxtimes C_{m_{k-1}})$ is equal to the theoretical minimum of $3^k + 1$, where C_i denotes the cycle of length i and $\boxtimes$ denotes the strong product of graphs.**

***Index Terms*—Cycle, frequency assignment, graph theory, $L(2, 1)$-labeling, λ-number, strong product.**

I. Introduction

CONSIDER the problem of assigning frequencies to radio transmitters at various nodes in a territory. The transmitters that are close must receive frequencies that are sufficiently apart, for otherwise they may be at the risk of interfering with each other. The spectrum of frequencies is an important resource on which there are increasing demands due to modern communication needs, both civil and military. This calls for an efficient management of the spectrum. It is assumed that the transmitters are identical and the signal propagation is isotropic.

The foregoing problem, with the objective of minimizing the span of frequencies, was first placed on a graph-theoretical footing in 1980 by Hale [1] who established its equivalence to the generalized vertex coloring problem, that is known to be computationally hard. (Vertices correspond to transmitter locations and their labels to radio frequencies, while adjacencies are determined by geographical "closeness" of the transmitters.)

Roberts [2] subsequently proposed a variation to the problem in which distinction is made between transmitters that are "close," and those that are "very close." This enabled Griggs and Yeh [3] to for-

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mulate the $L(2, 1)$-*labeling* of graphs that has since been an object of extensive research [4]–[9].

Formally, an $L(2, 1)$-labeling of a graph G is an assignment f of nonnegative integers to the vertices of G such that

- $|f(u) - f(v)| \geq 2$ if $d(u, v) = 1$
- $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$.

The difference between the largest label and the smallest label assigned by f is called the span of f, and the minimum span over all $L(2, 1)$-labelings of G is called the λ-*number* of G, denoted by $\lambda(G)$. The general problem of determining $\lambda(G)$ is NP-hard [4]. The following result consists of a useful lower bound.

Lemma 1.1 (Griggs and Yeh [3]): Let G be a graph with maximum degree $\Delta \geq 2$. If G contains three vertices of degree Δ such that one of them is adjacent to the other two, then $\lambda(G) \geq \Delta + 2$.

The lower bound of Lemma 1.1 is achievable in many cases [5], [6], [9].

By a graph is meant a finite, simple, undirected and connected graph. The *strong product* $G \boxtimes H$ of graphs $G = (V, E)$ and $H = (W, F)$ is defined as follows:

$$V(G \boxtimes H) = V \times W$$

$$E(G \boxtimes H = \{\{(u, x), (v, y)\}: u = v \text{ and } \{x, y\} \in F$$
$$\text{or } \{u, v\} \in E \text{ and } x = y$$
$$\text{or } \{u, v\} \in E \text{ and } \{x, y\} \in F\}.$$

Note, that $|V(G \boxtimes H)| = |V| \cdot |W|$ and $|E(G \boxtimes H)| = |V| \cdot |F| + |W| \cdot |E| + 2 \cdot |E| \cdot |F|$. This product (that is commutative and associative in a natural way) is one of the most important graph products. A particular application may be seen in the area of Shannon capacity of a memoryless channel [10]. It is easy to see that $G \boxtimes H$ is connected iff G and H are connected. Also, $d_{G \boxtimes H}((u, x), (v, y)) = \max\{d_G(u, v), d_H(x, y)\}$ [11]. It is further relevant to note that $\boxtimes$-product of finitely many cycles is edge-decomposable into Hamiltonian cycles [12]. Accordingly, it has high fault tolerance with respect to node failure and edge failure.

For $m \geq 3$ and $n \geq 2$, let C_m denote the *cycle* on m vertices, and let P_n denote the *path* on n vertices, where $V(C_k) = V(P_k) = \{0, \ldots, k-1\}$, $E(P_k) = \{\{i, i+1\}: 0 \leq i \leq k-2\}$ and $E(C_k) = E(P_k) \cup \{\{k-1, 0\}\}$. The graph $P_5 \boxtimes P_4$ appears in Fig. 1, where vertices (p, q) have been shown as pq.

Consider the graph $C_{m_0} \boxtimes \ldots \boxtimes C_{m_{k-1}}$. Vertices adjacent to $(v_0, \ldots, v_{k-1})$ are of the form $(v_0 + a_0, \ldots, v_{k-1} + a_{k-1})$ where $a_0, \ldots, a_{k-1}$ are not all zero, $a_i \in \{+1, 0, -1\}$, and $v_i + a_i$ is modulo m_i, where $0 \leq i \leq k-1$. It follows that $C_{m_0} \boxtimes \ldots \boxtimes C_{m_{k-1}}$ is a regular graph of degree $3^k - 1$. By Lemma 1.1, $\lambda(C_{m_0} \boxtimes \ldots \boxtimes C_{m_{k-1}}) \geq 3^k + 1$. The central result of this paper is that the preceding lower bound is achieved if $m_0, \ldots, m_{k-1}$ are each a multiple of $3^k + 2$. An analogous result was recently obtained by the author with respect to the Cartesian product [9].

Among other things, the following fact will be used in the sequel: If a, b and n are integers such that n is positive and $0 < |a - b| < n$, then $|(a \bmod n) - (b \bmod n)| = |a - b|$ or $n - |a - b|$.

II. Result

Theorem 2.1: If $k \geq 1$ and $m_0, \ldots, m_{k-1}$ are each a multiple of $3^k + 2$, then $\lambda(C_{m_0} \boxtimes \ldots \boxtimes C_{m_{k-1}}) = 3^k + 1$.

Proof: For $k = 1$, there is a single cycle C_{5j} for which the claim is known to be true [7]. In what follows, let $k \geq 2$, and let G denote $C_{m_0} \boxtimes \ldots \boxtimes C_{m_{k-1}}$. By Lemma 1.1, $\lambda(G) \geq 3^k + 1$, so it suffices to present an $L(2, 1)$-labeling of G using the labels $0, \ldots, 3^k + 1$. Let $n = 3^k + 2$.

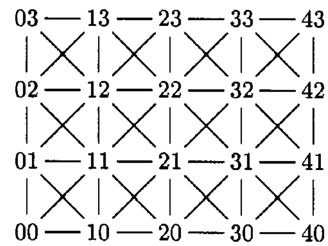

Fig. 1. The graph $P_5 \boxtimes P_4$.

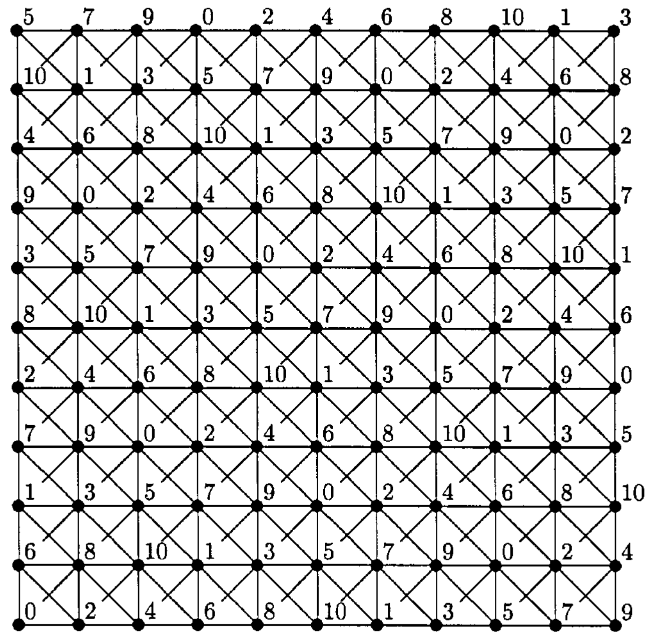

Fig. 2. $L(2, 1)$-labeling of $P_{11} \boxtimes P_{11}$ toward that of $C_{11} \boxtimes C_{11}$.

Let a vertex $v = (v_0, \ldots, v_{k-1})$ be assigned the integer

$$f(v) = \left[\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot v_i\right] \bmod n.$$

The assignment is clearly well-defined.

Vertices adjacent to $(v_0, \ldots, v_{k-1})$ are of the form $(v_0 + a_0, \ldots, v_{k-1} + a_{k-1})$, where $a_0, \ldots, a_{k-1}$ are not all zero, $a_i \in \{+1, 0, -1\}$, $0 \leq i \leq k-1$, and $v_i + a_i$ is modulo m_i. It is clear that vertex $w = (v_0 + a_0, \ldots, v_{k-1} + a_{k-1})$ receives the label

$$f(w) = \left[\left(\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot v_i\right) + \left(\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot a_i\right)\right] \bmod n.$$

Note that

- $(\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot a_i)$ is an even integer;
- Letting r be the largest integer such that $a_r \neq 0$, $|(\sum_{i=0}^{r-1} 2 \cdot 3^i \cdot a_i)| < 2 \cdot 3^r \cdot |a_r|$, so $(\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot a_i)$ is of the same sign as a_r, hence $(\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot a_i) \neq 0$;
- $|(\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot a_i)| \leq (\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot |a_i|) \leq (\sum_{i=0}^{k-1} 2 \cdot 3^i) = 3^k - 1 = n - 3$.

Letting $s = (\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot a_i)$, it is clear that $f(v)$ is of the form $N \bmod n$ and $f(w)$ is of the form $(N + s) \bmod n$. Since $2 \leq |s| \leq n - 3$, it is easy to see that $|f(v) - f(w)| = |s|$ or $n - |s|$, each of which is between 2 and $n - 2$. It follows that adjacent vertices receive labels that differ by at least two.

Vertices at a distance of two from $(v_0, \ldots, v_{k-1})$ are of the form $(v_0 + b_0, \ldots, v_{k-1} + b_{k-1})$, where $b_0, \ldots, b_{k-1}$ are not all different from $+2$ or -2; $b_i \in \{+2, +1, 0, -1, -2\}$ and $v_i + b_i$ is modulo m_i. Vertex $(v_0 + b_0, \ldots, v_{k-1} + b_{k-1})$ receives the label

$$\left[\left(\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot v_i\right) + \left(\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot b_i\right)\right] \bmod n.$$

Note that

- $(\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot b_i)$ is an even integer;
- Letting r be the largest integer such that $b_r \neq 0$, $|(\sum_{i=0}^{r-1} 2 \cdot 3^i \cdot b_i)| \leq (\sum_{i=0}^{r-1} 2 \cdot 3^i \cdot |b_i|) \leq \sum_{i=0}^{r-1} 4 \cdot 3^i = 2 \cdot (3^r - 1) < 2 \cdot 3^r \cdot |b_r|$, so $(\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot b_i)$ is of the same sign as b_r, hence $(\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot b_i) \neq 0$;
- $|(\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot b_i)| \leq (\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot |b_i|) \leq (\sum_{i=0}^{k-1} 4 \cdot 3^i) = 2 \cdot (3^k - 1) < 2n$.

Since n itself is odd, $(\sum_{i=0}^{k-1} 2 \cdot 3^i \cdot b_i)$ is not a multiple of n. It follows that vertices that are at a distance of two from each other receive different labels.

Conclusions are valid even if v_i is of the form $m_i - 2$ or $m_i - 1$, since each m_i is itself a multiple of n, and the arithmetic is modulo n. ■

Example: For $k = 2$, the $L(2, 1)$-labeling of $C_{11} \boxtimes C_{11}$ based on the proof of Theorem 2.1 is illustrated in Fig. 2 by means of the labeling of $P_{11} \boxtimes P_{11}$.

The following remarks are relevant with respect to the construction in the proof of Theorem 2.1.

2) Labels used are consecutive. Not all graphs admit consecutive (no-hole) $L(2, 1)$-labeling [6].
3) If a vertex v receives the label i, then those adjacent to v receive labels from $\{0, \ldots, 3^k + 1\} \setminus \{i - 1, i + 1\}$, where $i - 1$ and $i + 1$ are modulo $3^k + 2$.
4) If G is a subgraph of $C_{m_0} \boxtimes \ldots \boxtimes C_{m_{k-1}}$ such that the largest degree of a vertex of G is equal to the degree of a vertex of $C_{m_0} \boxtimes \ldots \boxtimes C_{m_{k-1}}$ and G satisfies Lemma 1.1, then $\lambda(G) = 3^k + 1$.
5) Let V_i denote the set of vertices of $C_{m_0} \boxtimes \ldots \boxtimes C_{m_{k-1}}$ that receive label i, where $0 \leq i \leq 3^k + 1$.
   b) $V_0, \ldots, V_{3^k+1}$ form a vertex partition of $C_{m_0} \boxtimes \ldots \boxtimes C_{m_{k-1}}$ into equal-size independent sets.
   c) Each V_i dominates a total of $(3^k/(3^k + 2)) \cdot |V|$ vertices (including those in V_i itself), where V denotes the vertex set of the graph.
   d) Elements of each V_i correspond to as many vertex-disjoint $K_{1, 3^k - 1}$s.
   e) For $0 \leq j \leq ((3^k - 1)/2)$, elements of each $(V_{2j} \cup V_{2j+1})$ correspond to as many edge-disjoint $K_{1, 3^k - 1}$s.

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ftd: Frequency to Time Domain Conversion for Reduced-Order Interconnect Simulation

Ying Liu, Lawrence T Pileggi, and Andrzej J Strojwas

***Abstract*—Model order reduction is an invaluable tool for solving part of the interconnect simulation problem, but an equally important problem is to interface these frequency-domain representations with the discrete time domain models that are used for nonlinear transient simulation. Various recursive convolution methods have been proposed which provide an exact solution to this interface problem under the assumption of piecewise linear voltage waveforms. This paper proposes a frequency to time-domain conversion algorithm, *ftd* (frequency to time domain), that represents a seemingly optimal form of recursive convolution. We will demonstrate that *ftd* provides exact accuracy for piecewise linear inputs. Moreover, *ftd* is algorithmically straight-forward and simple to implement for a specific application.**

***Index Terms*—Circuit simulation, interconnect, recursive convolution.**

I. INTRODUCTION

A DIGITAL VLSI circuit model can be partitioned into two subcircuits: nonlinear and linear. The nonlinear subcircuit consists of transistors, while the linear subcircuit consists of the interconnect. For today's CMOS technologies, the linear interconnect can dominate the circuit path delay. For efficiency, and due to the enormous size of the interconnect equivalent-circuits, model-reduction techniques are often applied to pre-process the linear interconnect blocks into lower-order N-port approximations[8]. In order to calculate delay or waveform, these compact models have to be incorporated into a circuit/timing simulator

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